

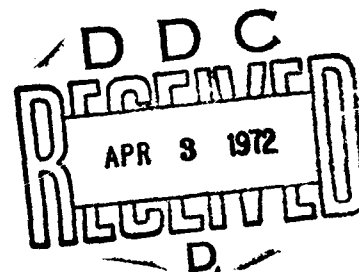
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**STATISTICAL ANALYSIS TECHNIQUES FOR ERROR PROPAGATION
IN LARGE-SCALE MISSILE SYSTEMS**

by
James R. Rowland and Willard M. Holmes

August 1971



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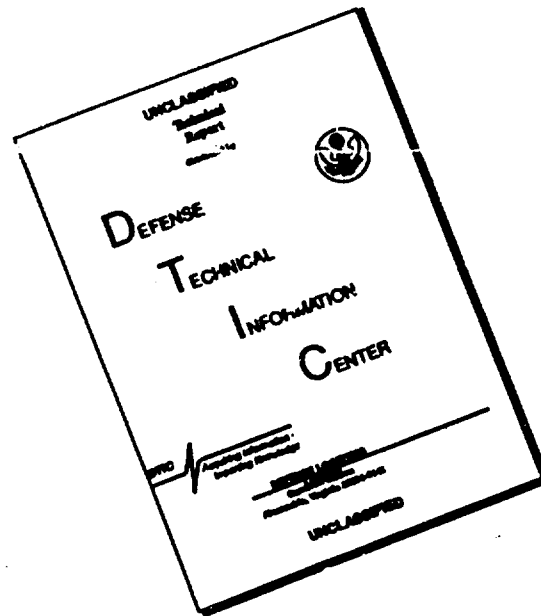
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Redstone Arsenal, Alabama

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Guidance and Control Directorate
Directorate for Research, Development, Engineering
and Missile Systems Laboratory
U.S. Army Missile Command
Redstone Arsenal, Alabama 35809

ABSTRACT

The statistical covariance technique is proposed as a computer analysis tool for missile systems error propagation studies. It is shown that on representative examples the statistical covariance technique performs better, based on speed and accuracy, than the traditional Monte Carlo approach of averaging a large number of random simulation runs. Although exact only for linear, time-varying systems, the covariance technique gives attractive approximate results for a wide range of nonlinear missile systems. The extension to nonlinear systems utilizes incremental equations about the noise-free solution. Except in the extreme cases of very harsh nonlinearities and high input noise levels, the statistical covariance technique has been shown to yield, for the systems considered, results comparable to 1000 Monte Carlo runs. Moreover, an accuracy prediction procedure has been developed to estimate the error to be expected from statistical covariance as well as the number of Monte Carlo runs required for comparable accuracy. Furthermore, an automatic sensitivity program is suggested for utilizing the statistical covariance technique on existing simulations without reprogramming. It is shown that the usual number of Monte Carlo runs yields very inaccurate results, e.g., for one system considered 25 runs results in an error of 30 percent while 100 runs gives 10 percent error. A total computer software package for statistical covariance, including automatic sensitivity, accuracy prediction, and miss distance programs, is proposed for further development.

CONTENTS

	Page
1. Introduction	1
2. Monte Carlo Studies	4
3. The Statistical Covariance Technique	15
4. Program Refinements for the Statistical Covariance Technique	40
5. Sequential Calculations of the Statistical Covariance Equations	44
6. Conclusions and Recommendations	48
Appendix A. Computer Programs for Pseudo-Random Number Generation	53
Appendix B. The Exact Convolution Solution for the Output Variance in Linear Systems	61
Appendix C. Combined Statistical Covariance and Monte Carlo Computer Programs	63
Appendix D. A Transformation for Normal Random Variables	67
References	69

ILLUSTRATIONS

Figure		Page
1	Multiplicative Pseudo-Random Number Brown ($M = 2^{20}$) Generator	6
2	Sample Variances for Two Multiplicative Generators	7
3	Sample Means for Two Multiplicative Generators	8
4	Discretization Effects for White Noise	9
5	Monte Carlo Results for the System in Equation (7) Using the Brown ($M = 2^{20}$) Generator with the Exact Nonlinear Transformation to Normal	11
6	Average Percent Error on Output Variance Versus the Number of Monte Carlo Runs for Generators Using the Exact Nonlinear Transformation in Equation (4)	12
7	Average Percent Error on Output Variance Versus the Number of Monte Carlo Runs for Generators Using the Central Limit Transformation	13
8	Determining the Optimum Utilization of 100,000 Random Numbers in Monte Carlo Simulations for System in Equation (7)	15
9	A Sketch Showing Incremental Variations $\delta \underline{x}(t)$ About a Nominal Trajectory $\underline{x}_N(t)$	20
10	The Open-Loop System and System Nonlinearity Described by Equation (24)	22
11	Monte Carlo Sample Functions and Statistical Covariance Results	23
12	Variations in Statistical Covariance Results as a Function of the Amount of System Nonlinearity	24
13	Variations in Statistical Covariance Results as a Function of the Input Noise	25
14	Monte Carlo and Statistical Covariance Results Versus Time	26
15	Variations in Average Percent Error in the Output Variance Versus Q_w	27
16	Time Plots of the Output Variance for Various $x_2(0)$	28
17	Closed-Loop System Description for Equation (25)	29
18	Monte Carlo and Statistical Covariance Results Versus Q_w for the Closed-Loop System for Equation (25)	30
19	Plots of k Versus α and Q_w	32
20	Plots of Percent Error for Statistical Covariance Versus k	33
21	The Incremental Variation in the Miss Distance Resolved Into Components	36
22	The Rayleigh Probability Density Function	38
23	The Basic Large-Scale System Showing Individual Subsystem Connections	45
24	A Flow Chart of the Sequential Algorithm	49

1. Introduction

a. Noise in Missile Systems

The consideration of noise problems is an inherent part of the analysis and design of large-scale missile systems. The usual noise problems occur when either system parameters or disturbance inputs vary randomly. Such stochastic variations may be represented as random processes or, in the case of unknown bias effects, as random variables. Random disturbances often appear as noisy signal inputs or as measurement noise. Typical noise sources in missile systems are radar clutter, gyro drift and bias, boresight and alignment errors, and wind gust disturbances. In all cases, random variations in system parameters and external inputs result in errors which are propagated throughout the missile system.

This report examines the problem of noise propagation in large-scale missile systems from two basic viewpoints. The first of these is the Monte Carlo technique, which exercises a computer simulation of the missile system with a large number of random sample functions. The second approach is an analytical method referred to as the statistical covariance technique. Numerical results indicate that the statistical covariance approach is an effective and efficient tool for most missile systems analysis situations and, in most practical instances, a significant improvement over the Monte Carlo method.

b. Background on the Monte Carlo Method

Previously, the primary capability for Monte Carlo noise studies was based on the use of analog noise generators. Analog sources of noise generally depend upon a gas tube in a magnetic field. Bekey and Karplus [1] have argued that the specifications of analog noise generators are usually only approximately known and that better noise sources are available. Other noise studies rely on tables of random numbers [2], but the use of tables requires some type of storage device. More recently, digital pseudo-random number generators have become the standard input for Monte Carlo simulations. Chambers [3], Hull and Dobell [4], MacLaren and Marsaglia [5], and Gelder [6] are among those who have developed mixed congruential and multiplicative recurrence formulas for generating pseudo-random numbers.* These formulas yield numbers that are uniformly distributed on the unit interval (0, 1).

*The term pseudo-random is appropriate because the numbers are not actually random but are generated by deterministic means. This deterministic origination has the inherent advantage of providing reproducible input sequences which are often useful for computer program debugging.

The problem of generating digital representations of correlated continuous random inputs has been treated in a report by Rowland [7]. Techniques for designing digital shaping filters have been developed in reports by Gujar and Kavanagh [8], Holliday [9], Baum [10], Broste [11], and Conner [12]. Moreover, digital simulation problems encountered when using only a finite sequence of random data are discussed in reports by Jansson [13] and Brown and Rowland [14]. A comprehensive study resulting in a computer software package for Monte Carlo simulations was performed within the U.S. Army Missile Command, Redstone Arsenal by J. L. Harris [15].

It should be emphasized that any statistical information related to noise propagation in missile systems simulations must agree with correctly implemented Monte Carlo results.

c. Background for the Statistical Covariance Technique

The statistical covariance approach stems from the use of the error covariance matrix in the Kalman filtering equations [16, 17]. Almost every new book in the last five years on stochastic estimation and control includes a discussion of the statistical covariance technique for linear, time-varying systems [18-23]. The method has also been applied for handling variations about a nominal flight path obtained from nonlinear system equations. Kuhnel and Sage [24] applied the covariance technique to sensitivity equations about the nominal flight path caused by trajectory initial dispersions and parameter variations. They considered a thirty-third order, 6 degree-of-freedom homing missile model to illustrate the usefulness of the technique to a realistic situation. Irwin and Hung [25] applied the same method to nonlinear systems having random bias inputs, e.g., misalignment or drift errors. They compared the direct and adjoint approaches for evaluating the state covariance matrix, while Kuhnel and Sage used only the adjoint method. These papers demonstrated that the covariance technique is useful for nonlinear systems. There have been other proposed linearization schemes which yielded varying degrees of success [26, 27]. These approaches are based on component by component linearization rather than the more powerful approach of linearizing variational equations about the exact nominal trajectory. More recently, Brown [28-30] has used the statistical covariance approach to solve trajectory optimization problems for nonlinear feedback systems. In addition, Clark [31, 32] has developed related results using improved estimation about a nominal trajectory.

The statistical covariance technique is generally recognized as the modern approach for handling error propagation in guidance and control systems. Its application to missile systems analysis, emphasizing advantages and limitations, will be demonstrated in later paragraphs of this report.

d. The Error Propagation Problem

The analysis problem to be considered here may be described by the vector differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{r}(t), \underline{w}(t), t) \quad , \quad (1)$$

where

\underline{x} = the n-dimensional vector of system variables

$\underline{r}(t)$ = the non-random inputs

$\underline{w}(t)$ = the continuous random process disturbance inputs

t = the independent variable representing time.

For simplicity of notation, the vector $\underline{r}(t)$ in Equation (1) will be suppressed hereafter, and the system under consideration will be expressed as

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{w}, t) \quad . \quad (2)$$

The problem is to determine the probability density function of the system variables \underline{x} as a function of time t . In most cases of interest, it will be sufficient to find only the mean and variance of $\underline{x}(t)$.

e. Outline of the Report

Following this brief introductory background material and problem definition, Paragraph 2 describes the Monte Carlo approach in detail and gives numerical results for a simple example. It is shown that a very large number of simulation runs (up to 1000) are needed to yield acceptable accuracy. The statistical covariance technique is introduced in Paragraph 3. The basic approach is developed for linear systems and extended for an approximate analysis of nonlinear systems. Paragraph 4 presents program refinements, including an automatic interrogation program for determining sensitivity coefficients in existing large-scale missile simulations. Paragraph 5 discusses the use of the statistical covariance technique for large-scale systems and an approximate sequential algorithm is developed. Finally, Paragraph 6 summarizes the conclusions of this report and indicates problem areas for future consideration.

2. Monte Carlo Studies

The traditional approach for the analysis of noise propagation in dynamical systems involves Monte Carlo-based statistics. Two elements are needed in the digital implementation of the Monte Carlo method. The first of these is the digital generation of a sequence of pseudo-random numbers for input to the given system. Several multiplicative random number generators are investigated in this paragraph. The second consideration is the sampling problem inherent in representing continuous systems and signals digitally. The modification of the variance of the generated random sequence as a function of the integration step size is necessary to handle this sampling problem.

This paragraph selects a particular digital random number generator for use in subsequent work by testing several popular generators on a representative system. It should be noted that even this "best" generator yields poor results when only 100 to 200 Monte Carlo runs are used. Therefore, the case for implementing the statistical covariance technique to be described in Paragraph 3 is strengthened.

a. Digital Random Number Generators

The recurrence formula often utilized in generating pseudo-random numbers has the form

$$Z_{k+1} = AZ_k \quad (\text{Modulo } M) \quad , \quad (3)$$

which is referred to as the multiplicative method. The scalar constants A and M may be selected to insure good statistical properties. Several requirements for selecting A and M are given in a report by Brown and Rowland [14], from which the values of $A = 19971$ and $M = 2^{20}$ are highly recommended for use with a starting number $Z_0 = 31571$. It should be noted that the suggested generator yielded pseudo-random numbers with considerably better statistical properties than several other widely used generators, including the Harris generator [15].

The meaning of the modulo operation in Equation (3) is that the product AZ_k is first divided by M and the remainder is taken as Z_{k+1} , the next random number. Each random number may be normalized to the unit interval by dividing by M. The resulting sequence of numbers will be approximately uniformly distributed on (0, 1). Moreover, the random number sequence satisfies the requirements necessary for the digital representation of Gaussian white noise.

Box and Muller [33] developed a direct (or exact) approach for transforming two independent random variables from the same uniform distribution on the unit interval to a pair of independent random variables obeying a zero-mean, unity-variance normal distribution. The uniform random variables Z_1 and Z_2 are related to the normal random variables Y_1 and Y_2 by

$$\begin{aligned} Y_1 &= \left(-2 \log_e Z_1 \right)^{\frac{1}{2}} \cos 2\pi Z_2 \\ Y_2 &= \left(-2 \log_e Z_1 \right)^{\frac{1}{2}} \sin 2\pi Z_2 \end{aligned} \quad (4)$$

These transformations were used in obtaining some of the numerical results to be presented. A second method for obtaining normally distributed random numbers is to sum 12 numbers uniformly distributed on (0, 1) and subtract the sum of their means. This approach, referred to as the central-limit technique, is easier to apply but requires a larger number of random numbers for averaging. Both methods are used on a second-order system later in the following section.

b. Numerical Results

Two multiplicative digital pseudo-random generators were tested initially on the XDS Sigma 5. Both used $A = 19971$ with a starting number $Z_0 = 31571$. The program for the first generator, for which $M = 2^{20} = 1048576$ as suggested by Brown [14], is shown in Figure 1. Program statements 4 through 7 provide initialization for the uniform generator given by statements 13 through 21 and 28. Statements 25 and 26 transform the uniform results into a normal distribution according to Equation (4). The first 30 numbers from this generator are given in Figure A-1 of Appendix A. Unnormalized integers IX, the uniformly distributed numbers U, and the normally distributed numbers XNORM are listed.

The second generator differs from the first only in that $M = 2^{31}$ is used. To effect this change, statements 14 through 21 of the program in Figure 1 were replaced by the four statements

$$\begin{aligned} IX &= IY \\ IF(IX) \ 5, \ 5, \ 6 \\ 5 \ IY &= IY + 2147483647+1 \\ 6 \ U &= IY * 0.4656612873E-9 \end{aligned} \quad (5)$$

```

1      C      OPERATION OF A MULTIPLICATIVE DIGITAL GENERATOR
2      C
3      C
4      XMEAN=0.
5      SIG=1.
6      IX=31571
7      DUM=0.1
8      DO 99 K=1,30
9      C
10     C      MULTIPLICATIVE GENERATOR
11     C      WITH M=2 TO THE 31ST POWER
12     C
13     IY=19971*IX
14     IX=IY
15     IF(IX)5,5,6
16     5      IY=IY+2147483647+1
17     6      U=IY*0.4656612873E-9
18     C
19     C      TRANSFORMATION TO NORMAL
20     C
21     Z=SQRT(-2.0*ALOG(DUM))*SIG
22     XNORM=Z*COS(6.28318*U)+XMEAN
23     C
24     DUM=U
25     PRINT 77,K,IX,U,XNORM
26     77     FORMAT(6X,I4,8X,I12,2F15.6)
27     99     CONTINUE
28     STOP
29     END

```

Figure 1. The Multiplicative Pseudo-Random Number Brown
($M = 2^{20}$) Generator

Because the XDS Sigma 5 is a 32-bit machine, the division by M is handled automatically when M is selected as 2^{31} . Therefore, the resulting program, as evidenced by Equation (5), is simpler. The latter two statements in Equation (5) represent register shifts which accomplish the automatic division. Numerical results for the operation of the second generator are given in Figures A-2 and A-3 of Appendix A.

A program is given in Figure A-4 of Appendix A for examining some of the statistical properties of the two generators. Sample variance

and means are plotted in Figures 2 and 3 as functions of the number N of pseudo-random numbers generated. The following observations may be made regarding this data:

- 1) The prespecified means and variances are obtained as N becomes sufficiently large.
- 2) For finite sequences of generated numbers, the sample means and variances vary up to several percent about the prespecified values.

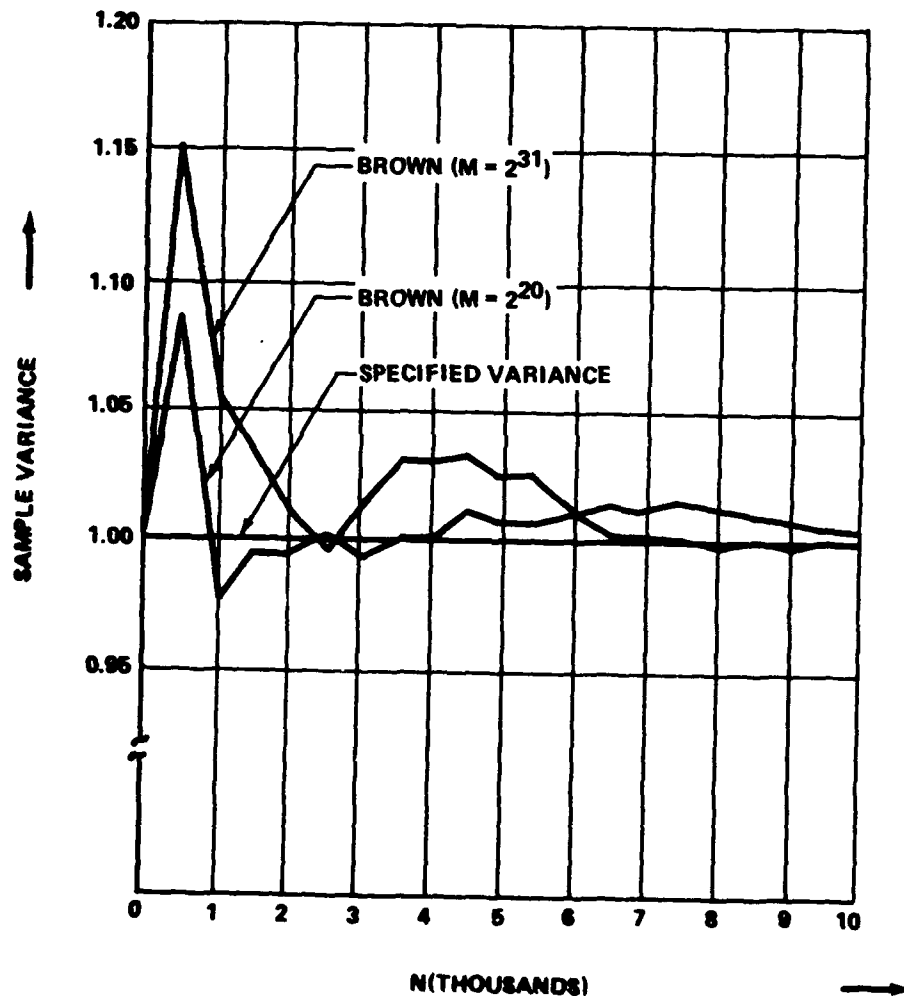


Figure 2. Sample Variances for Two Multiplicative Generators

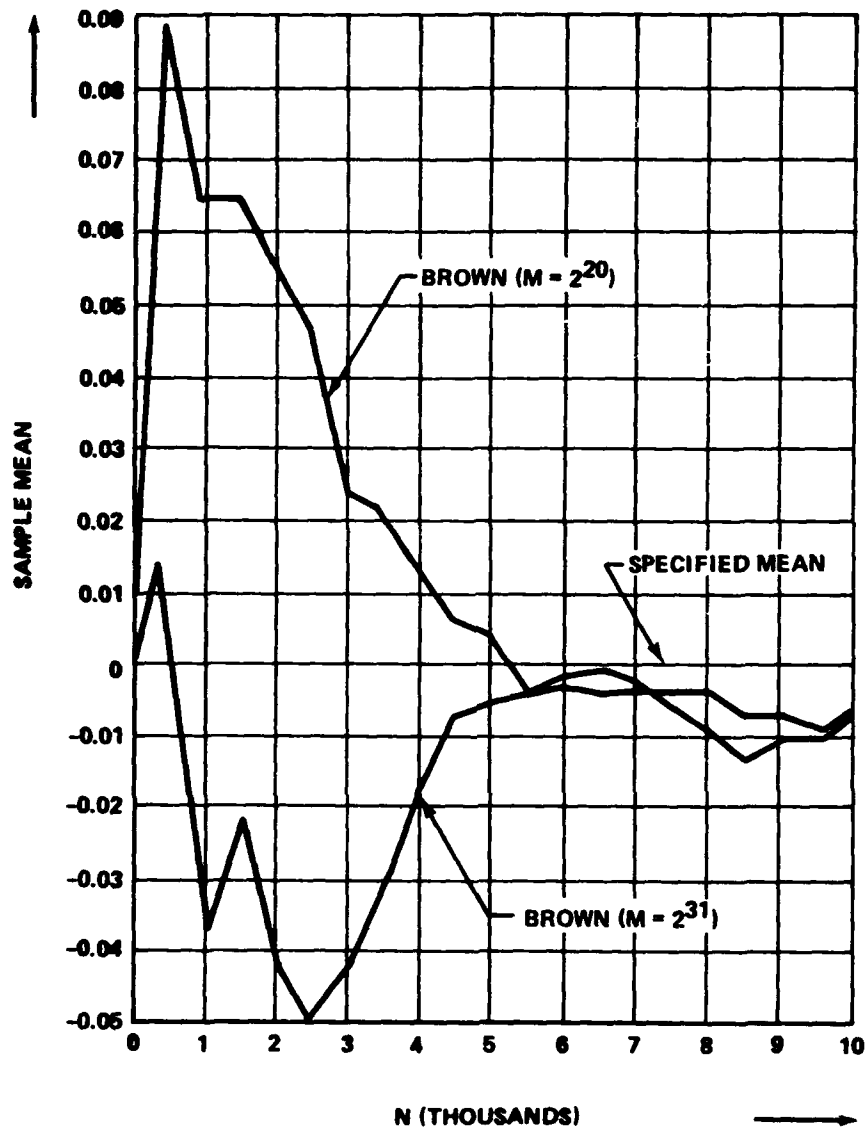


Figure 3. Sample Means for Two Multiplicative Genrators

Another important statistical property to be considered in random number generation is the autocorrelation function. It is noted in a report by Brown and Rowland [14] that the serial autocorrelation coefficients for independent Gaussian random sequences should be asymptotically normally distributed with mean zero and variance $1/N$. The Brown multiplicative generator with $M = 2^{20}$ satisfies the hypothesis of independence with a high degree of probability [14].

c. Discrete Representations for Continuous Noise Processes

The use of random numbers as disturbance input sample functions for dynamical systems requires special considerations. It is necessary to determine discrete representations for continuous noise processes [7]. If pseudo-random numbers are held constant over one sampling period H , then the corresponding autocorrelation function is the triangular form shown in Figure 4. It is required that this triangular function represent as closely as possible the impulse function for the continuous case. Equating the power spectral densities, i.e., Fourier transforms of the autocorrelation functions, of discrete and continuous cases, yields

$$Q_{wd} = \frac{Q_w}{H} \quad , \quad (6)$$

where Q_{wd} and Q_w are the variances of the discrete and continuous cases, respectively. Together with the proper utilization of a multiplicative pseudo-random number generator, the relationship in Equation (6) is the most important element in Monte Carlo simulations.

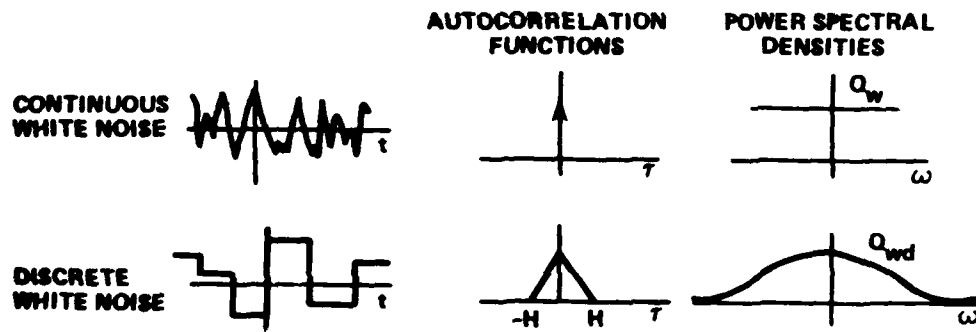


Figure 4. Discretization Effects for White Noise

d. Simulation Results

Consider the time-invariant, second-order, linear system described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + w \quad , \end{aligned} \quad (7)$$

with $x_1(0) = x_2(0) = 0$. Equation (7) gives, in phase variable form, the state equations of the system with transfer function

$$H(s) = \frac{1}{(s + 1)(s + 2)} \quad (8)$$

where x_1 represents the output x and x_2 represents \dot{x} . The input w is a zero-mean, unity-variance, gaussian white noise signal applied for all $t \geq 0$. The problem is to investigate the performance in Monte Carlo simulations for Equation (7) by using several multiplicative pseudo-random number generators.

The computer program used for testing the Brown ($M = 2^{20}$) generator with the exact nonlinear transformation in Equation (4) is included as Figure A-5 of Appendix A. The program shown was changed (statement 10) to allow 25, 50, 100, 200, 500, and 1000 simulation runs to be ensemble-averaged at half-second intervals from $t = 0$ to $t = 5$ seconds. The variance of the output x is plotted as a function of time t in Figure 5 for the Brown ($M = 2^{20}$) generator using the exact transformation in Equation (4). The exact variance solution is determined by convolution in Appendix B*. Large errors were obtained for all curves except the case using 1000 runs.

Figures 6 and 7 show the average percent error obtained for the output variance of Equation (7) as a function of the number of runs for several generators. Using the basic program in Figure A-5 in Appendix A, statements 27 through 34 were modified according to Equation (5) to give corresponding data for the Brown ($M = 2^{31}$) generator. Two new multiplicative generators were also tested. One of these, the SAM-D generator, has been used in guidance simulation studies for a large-scale air defense missile system. The SAM-D generator has a multiplier A of 899 and a starting number $Z_0 = 1073741823$. The other generator [34], which has been widely used on the IBM 360, has a multiplier A and starting number $Z_0 = 1220703125$. Figure 6 shows results for these four generators obtained by using the exact nonlinear transformation in Equation (4) from uniform to normal. Figure 7 shows results using the central-limit approach approach of summing 12 numbers from the uniform generators. It should be noted that, while the Brown ($M = 2^{20}$) generator had an average error of only 1.5 percent for 1000 runs, the largest error obtained at the 10 equally-spaced test points was 2.7 percent. Therefore, results for even 1000 runs may be considered to be only marginally acceptable in some applications.

*The statistical covariance technique to be described in Paragraph 3 also yields the exact solution. Moreover, the statistical covariance technique is much easier to apply as well as being more amenable for machine computation than the convolution approach.

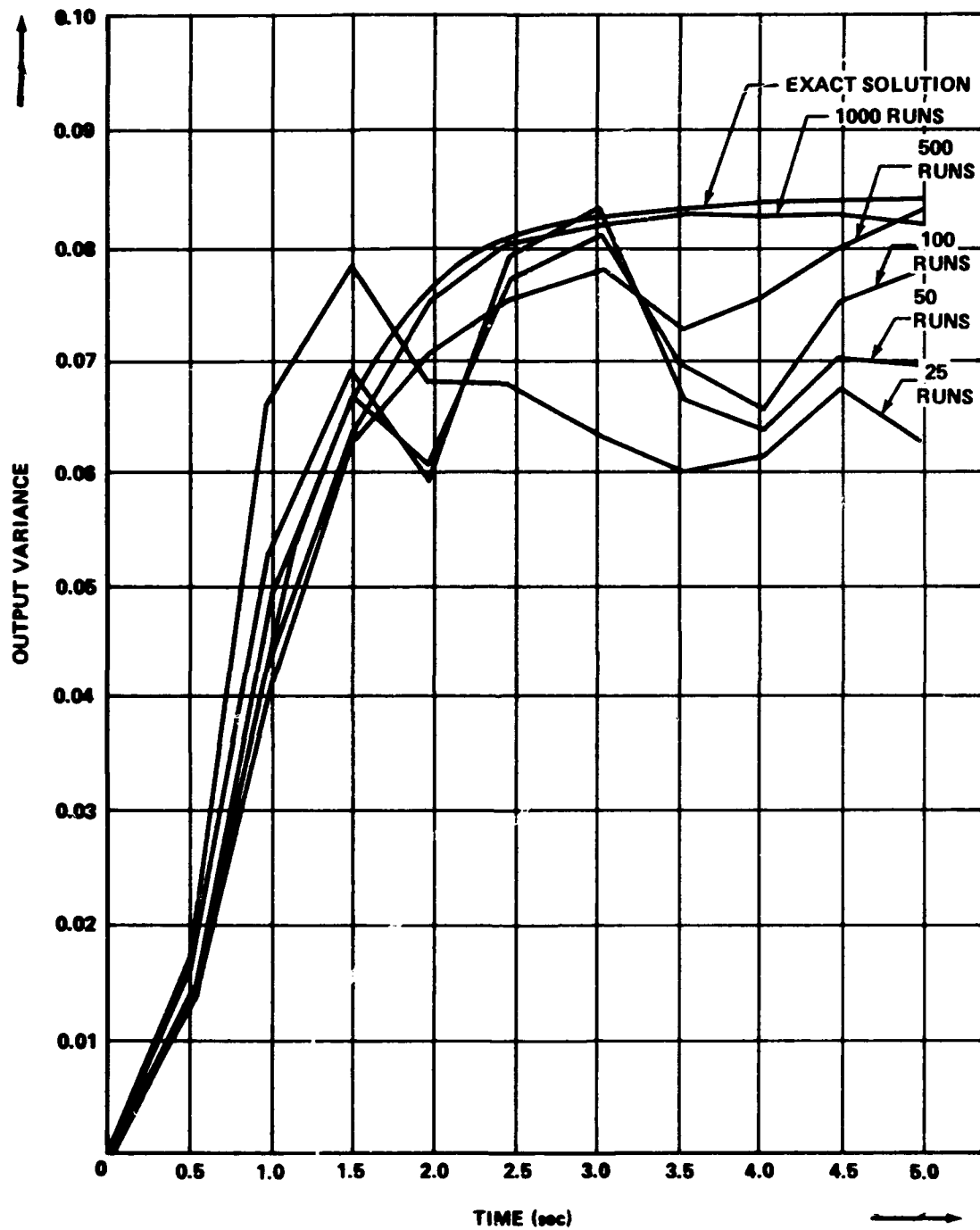


Figure 5. Monte Carlo Results for the System in Equation (7) Using the Brown ($M = 2^{20}$) Generator with the Exact Nonlinear Transformation to Normal

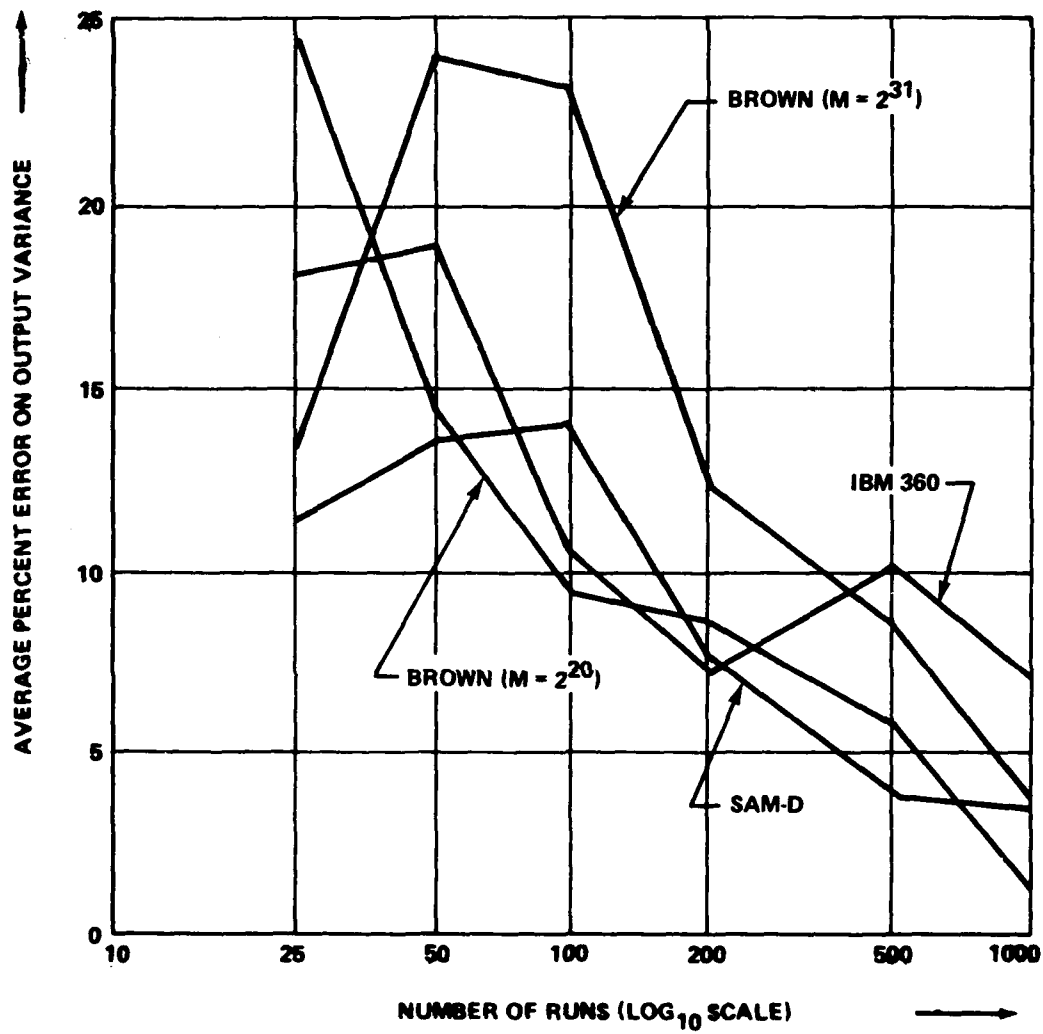


Figure 6. Average Percent Error on Output Variance Versus the Number of Monte Carlo Runs for Generators Using the Exact Nonlinear Transformation in Equation (4)

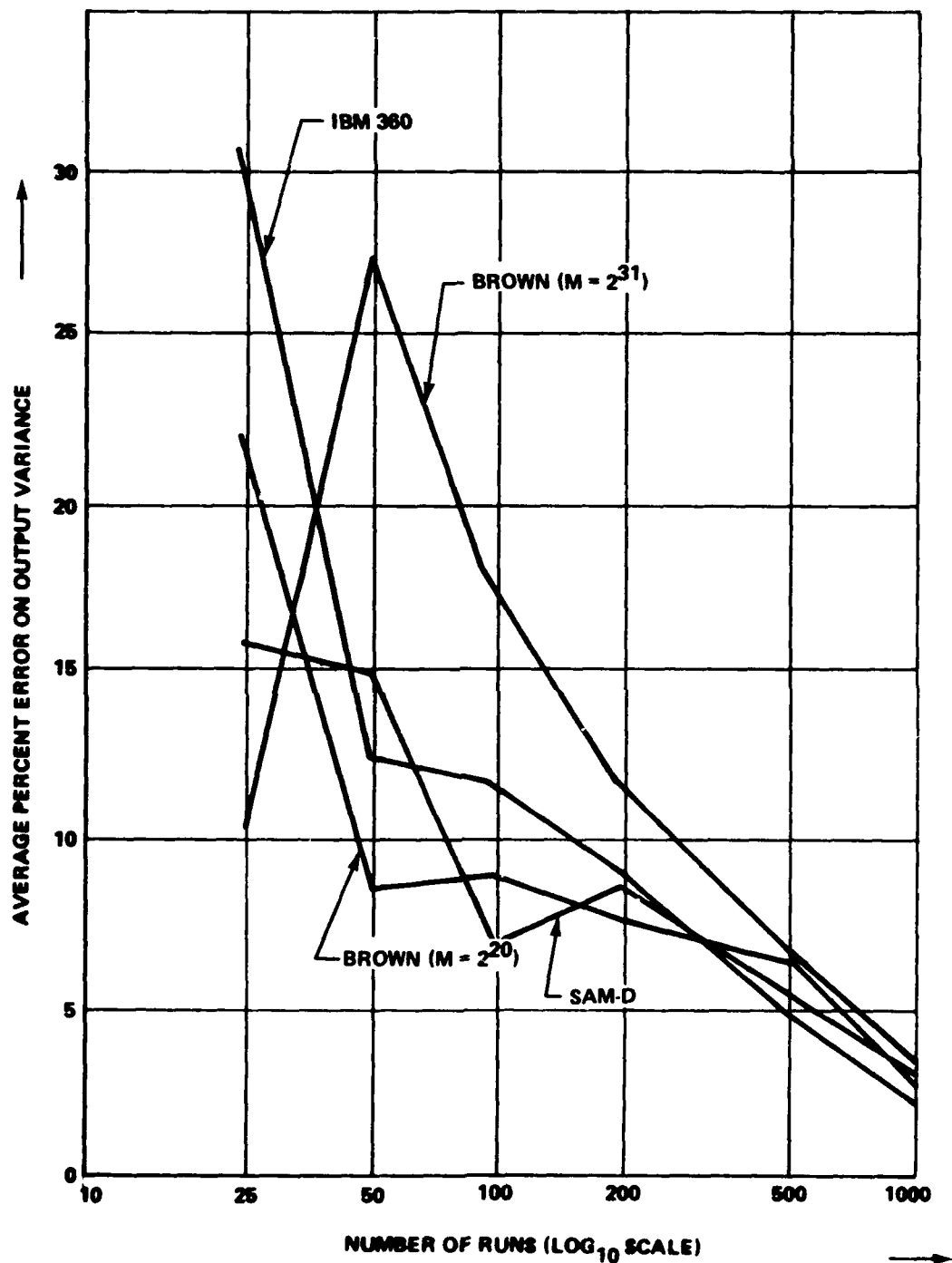


Figure 7. Average Percent Error on Output Variance Versus the Number of Monte Carlo Runs for Generators Using the Central-Limit Transformation

e. Step Sizes Versus Number of Runs

Consideration should also be given to the problem of how to use a fixed number of random numbers in a Monte Carlo simulation. For example, the question arises as to whether the error in the output variance would be smaller by using a large step size with many runs or a small step size with fewer runs. If N_p is the number of sample points per run and N_R is the number of runs to be ensemble-averaged, then the error in the output variance obviously decreases as either N_p or N_R increases. Note that the product $N_p N_R$ is the total number of random numbers to be used. The problem is to determine the values of N_p and N_R , when the product $N_p N_R$ is fixed, to yield the smallest possible error.

Figure 8 shows plots of average error for the output variance as a function of the step size H for the system in Equation (7) when the first 100,000 random numbers are used from the Brown ($M = 2^{20}$) generator. For example, the value at $H = 0.01$ for the 5-second problem was obtained by using $N_p = 500$ and $N_R = 200$. Two methods of averaging the error variance data at half-second intervals were used. The first treated all data points equally, but the second approach weighted the data according to the number of random numbers which had been put into the system prior to that particular problem time. For example, the data at $t = 3.0$ seconds was weighted with a factor of only 60 percent as large as the data at $t = 5.0$ seconds, because only 60,000 random numbers were applied to the system up to $t = 3.0$ seconds while 100,000 were used on the range $t = 0$ to 5.0 seconds. The curve in Figure 7 shows that as large a step size as possible should be used without violating conditions for which truncation errors become significant in integrating the system equations. In this example, the maximum step size under those conditions was $H = 0.5$ second. Note that the use of a somewhat smaller step size gave much larger average errors in the output variance.

f. Summary

A detailed description has been given on the digital implementation of the Monte Carlo method. Discretization problems and pseudo-random number generation have been discussed, and several multiplicative generators have been tested on a second-order linear system. Simulations showed that the Brown ($M = 2^{20}$) generator performed consistently well in a variety of tests. An interesting result is that for maximum efficiency in Monte Carlo simulations the largest step size compatible with integration accuracy should be used with a large number of computer runs for ensemble-averaging. Inaccuracies of the Monte Carlo method, as well as the large amount of computer time required, point to the need for the statistical covariance technique of Paragraph 3.

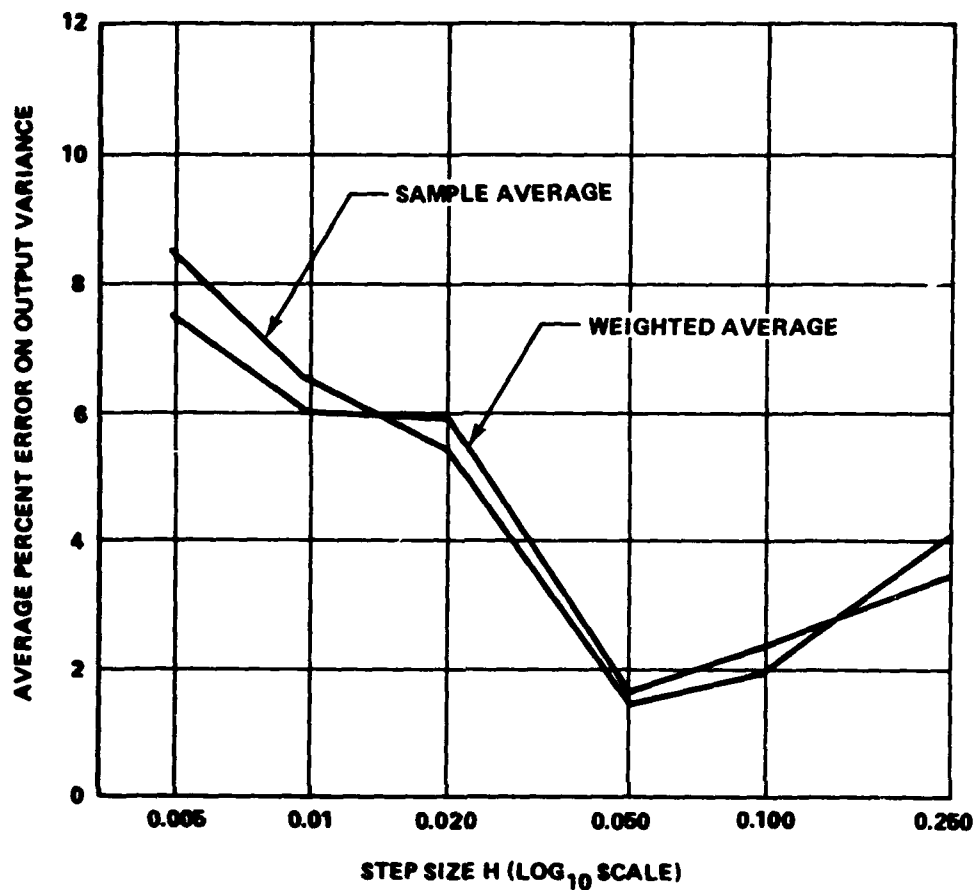


Figure 8. Determining the Optimum Utilization of 100,000 Random Numbers
in Monte Carlo Simulations for System in Equation (7)

3. The Statistical Covariance Technique

An analytical method is developed in this paragraph and compared on a nonlinear system with the Monte Carlo approach described in Paragraph 2. The statistical covariance technique is derived as an exact solution for linear, time-varying systems and then extended for an approximate analysis of the error propagation problem in nonlinear systems. The method is also applied to the problems of miss distance studies and

error budgeting. The primary conclusion of this paragraph is that the statistical covariance technique is valid for a wide range of input noise signals and parameters in nonlinear system analysis.

a. Derivation of Statistical Covariance Results

Let the system under consideration be described by a linear, time-varying vector differential equation of the form

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{w} \quad , \quad (9)$$

where

\underline{x} = the n-dimensional state vector

\underline{w} = an m-dimensional input vector

\underline{A} = an n by n matrix

\underline{B} = an n by m matrix .

The input vector \underline{w} has elements that are zero-mean white noise with a covariance matrix given by

$$E\{\underline{w}(t) \underline{w}^T(\tau)\} \triangleq \underline{Q}_w \delta(t - \tau) \quad . \quad (10)$$

The matrix \underline{Q}_w , as well as the matrices \underline{A} and \underline{B} , may be time-varying.

Let the covariance matrix of the state \underline{x} be defined by

$$P(t) \triangleq E\{\underline{x}(t) \underline{x}^T(t)\} \quad , \quad (11)$$

where T denotes the transpose.

The problem is to determine $P(t)$ in terms of \underline{A} , \underline{B} , and \underline{Q}_w . Three approaches to the problem will be used. The first two apply directly to the continuous system in Equation (9) and the third pertains to a discretized version of the system.

(1) The Integral Solution. The first derivation of an expression for $P(t)$ utilizes the integral solution for $\underline{x}(t)$. From Equation (9),

$$\underline{x}(t) = \Phi(t, t_0) \underline{x}(t_0) + \int_{t_0}^t \Phi(t, \tau) B(\tau) \underline{w}(\tau) d\tau \quad (12)$$

Therefore,

$$\begin{aligned} P(t) &= E\{\underline{x}(t) \underline{x}^T(t)\} \\ &= E\left\{\left[\Phi(t, t_0) \underline{x}(t_0) + \int_{t_0}^t \Phi(t, \tau) B(\tau) \underline{w}(\tau) d\tau\right] \right. \\ &\quad \left. \left[\Phi(t, t_0) \underline{x}(t_0) + \int_{t_0}^t \Phi(t, \tau) B(\tau) \underline{w}(\tau) d\tau\right]^T\right\} \quad (13) \end{aligned}$$

Performing the indicated multiplications in Equation (13) and noting that $\underline{x}(t_0)$ and $\underline{w}(t)$ are uncorrelated, yields

$$\begin{aligned} P(t) &= \Phi(t, t_0) E\{\underline{x}(t_0) \underline{x}^T(t_0)\} \Phi^T(t, t_0) \\ &\quad + \int_{t_0}^t \int_{t_0}^t \Phi(t, \tau_1) B(\tau_1) E\{\underline{w}(\tau_1) \underline{w}^T(\tau_2)\} \Phi^T(t, \tau_2) B^T(\tau_2) d\tau_1 d\tau_2 \quad (14) \end{aligned}$$

Using Equation (10) and the sifting property of the delta function, the following is obtained from Equation (14).

$$\begin{aligned} P(t) &= \Phi(t, t_0) P(t_0) \Phi^T(t, t_0) \\ &\quad + \int_{t_0}^t \Phi(t, \tau_1) B(\tau_1) Q_w(\tau_1) \Phi^T(t, \tau_1) B^T(\tau_1) d\tau_1 \quad (15) \end{aligned}$$

Equation (15) is the desired integral solution for $P(t)$. However, by forming $\dot{P}(t)$ from Equation (15) and using the relationship

$$\frac{d\Phi(t, \tau)}{dt} = A\Phi(t, \tau) \quad , \quad (16)$$

the result may be expressed as the following matrix differential equation:

$$\dot{P} = AP + PA^T + BQ_w B^T \quad . \quad (17)$$

This differential form of the statistical covariance equation is the primary result of Paragraph 2.

(2) The Differential Solution Directly. Using Equation (9) directly,

$$\begin{aligned} \dot{P} &= \frac{d}{dt} \left[E \left\{ \underline{x}(t) \underline{x}^T(t) \right\} \right] = E \left\{ \dot{\underline{x}} \underline{x}^T \right\} + E \left\{ \underline{x} \dot{\underline{x}}^T \right\} \\ &= E \left\{ (A\underline{x} + B\underline{w}) \underline{x}^T \right\} + E \left\{ \underline{x} (A\underline{x} + B\underline{w})^T \right\} \\ &= A E \left\{ \underline{x} \underline{x}^T \right\} + B E \left\{ \underline{w} \underline{x}^T \right\} + E \left\{ \underline{x} \underline{x}^T \right\} A^T + E \left\{ \underline{x} \underline{w}^T \right\} B^T \\ &= AP + PA^T + B E \left\{ \underline{w} \underline{x}^T \right\} + E \left\{ \underline{x} \underline{w}^T \right\} B^T \quad . \quad (18) \end{aligned}$$

Substituting Equation (12) into Equation (18) for $\underline{x}(t)$ and again applying the sifting property of the delta function to perform the integrations, Equation (17) is obtained [18].

(3) The Discrete Version. Equation (12) may be written in the discrete version as

$$\underline{x}_{k+1} = \Phi_k \underline{x}_k + h_k \underline{w}_k \quad . \quad (19)$$

Therefore,

$$\begin{aligned}
P_{k+1} &= E \left\{ \underline{x}_{k+1} \underline{x}_{k+1}^T \right\} \\
&= E \left\{ \left[\Phi_{k \rightarrow k} \underline{x}_k + h_k \underline{w}_k \right] \left[\Phi_{k \rightarrow k} \underline{x}_k + h_k \underline{w}_k \right]^T \right\} \\
&= \Phi_k P_k \Phi_k^T + h_k Q_{wd} h_k^T, \quad (20)
\end{aligned}$$

which is the discrete equivalent* of Equation (17).

b. Approximate Analysis of Nonlinear Systems

The application of the statistical covariance technique to nonlinear systems can be achieved as an approximate analysis. Consider small variations $\delta \underline{x}(t)$, caused by the input disturbance noise $\underline{w}(t)$, about the (noise-free) nominal trajectory $\underline{x}_N(t)$, as shown in Figure 9. These $\delta \underline{x}$ variations are assumed to be sufficiently small to satisfy linear perturbation equations, i.e.,

$$\underline{\delta x}(t) = \underline{x}(t) - \underline{x}_N(t) \quad (21)$$

and

$$\underline{\delta \dot{x}} = A(t) \underline{\delta x} + B(t) \underline{w}(t), \quad (22)$$

where

$$\begin{aligned}
A(t) &= \frac{\partial \underline{f}}{\partial \underline{x}} \bigg|_{\substack{\underline{x} = \underline{x}_N(t) \\ \underline{w} = \underline{w}}} \\
B(t) &= \frac{\partial \underline{f}}{\partial \underline{w}} \bigg|_{\substack{\underline{x} = \underline{x}_N(t) \\ \underline{w} = \underline{w}}} \quad (23)
\end{aligned}$$

*Recall that Q_w and Q_{wd} , the covariance matrices for the continuous and discrete cases, respectively, are related by $Q_{wd} = Q_w/H$, where H is the discretization interval as given in Equation (6) of Paragraph 2.

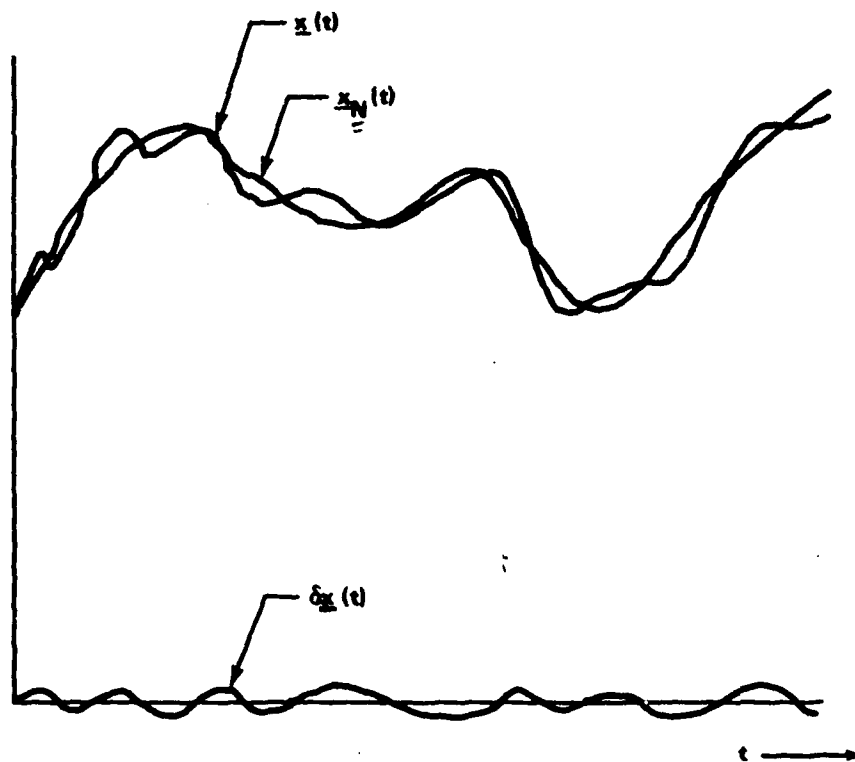


Figure 9. A Sketch Showing Incremental Variations $\delta \underline{x}(t)$ About a Nominal Trajectory $\underline{x}_N(t)$

\bar{w} denotes the mean of \underline{w} , (which has been assumed zero in the problem formulation). The nonlinear vector function \underline{f} is given in Equation (2). The only approximation made is that the second and higher order terms in $\delta \underline{x}$ are negligible when compared with the linear terms which appear in Equation (22). This approximation is valid when the $\delta \underline{x}$ variations are sufficiently small.

(1) Numerical Results for Nonlinear Systems. The following numerical results are given to provide insight into the practicality of using the statistical covariance technique for nonlinear systems. In particular, the ranges of noise signals and parameters of the system nonlinearity are determined for which the statistical covariance technique is recommended. Comparisons are made with 25, 50, 100, 200, and 1000 Monte Carlo runs.

(a) Example 1 - Consider the second-order nonlinear system shown in Figure 10. The system equations are

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + \alpha x_2^2 \operatorname{sign}(x_2) \\ \dot{x}_2 &= -x_2 + w(t) \quad ,\end{aligned}\tag{24}$$

where $w(t)$ is a Gaussian white noise process applied for $t \geq 0$. The process $w(t)$ has zero mean and variance Q_w . The initial condition on x_1 is zero, but $x_2(0)$ is allowed to vary in parts of this first example.

The statistical covariance technique and the Monte Carlo method were programmed for system Equation (24) in the single computer program given in Figure C-1 of Appendix C. The step size of $H = 0.05$ was selected according to the pseudo-random number utilization criterion developed in Paragraph 2. Note that for $\alpha = 0$ and $x_2(0) = 0$ the system in Figure 10 is identical to system Equations (7) and (8) of Paragraph 2, except that a different set of state variables has been chosen in Equation (24). Figure 11 shows a sketch of typical Monte Carlo runs along with the one-sigma contours from the statistical covariance results for $\alpha = 0.05$, $x_2(0) = 0.1$, and $Q_w = 0.05$. As the number of Monte Carlo runs becomes large, i.e., on the order of 1000 runs, the one-sigma contours from the statistical covariance technique are approximately the same (within 2 percent) as the one-sigma contours computed from the sample functions.

The validity of the statistical covariance technique for several values of α and Q_w is examined in Figures 12 and 13. In each case, the statistical covariance result agreed within 2 percent with 1000 Monte Carlo runs for small values of the parameters. As α and Q_w were increased, the error in the statistical covariance solution also increased. These curves are used later to estimate the accuracy expected from the statistical covariance technique by examining the nonlinear system equations directly.

Comparisons with a limited number of Monte Carlo runs are given in Figures 14 and 15. The first of these figures compares time plots of the output variance for 25, 50, 100, 200, and 1000 runs with the statistical covariance results. Figure 15 shows averaged results as percent error on the output variance versus the input noise covariance Q_w . All error curves are based on 1000 runs as the standard for comparison. In view of the results of Figures 6 and 7 in Paragraph 2, it should not be too surprising to find that even 200 runs yield errors of approximately 8 percent, while only 25 runs yield approximately 25 percent error.

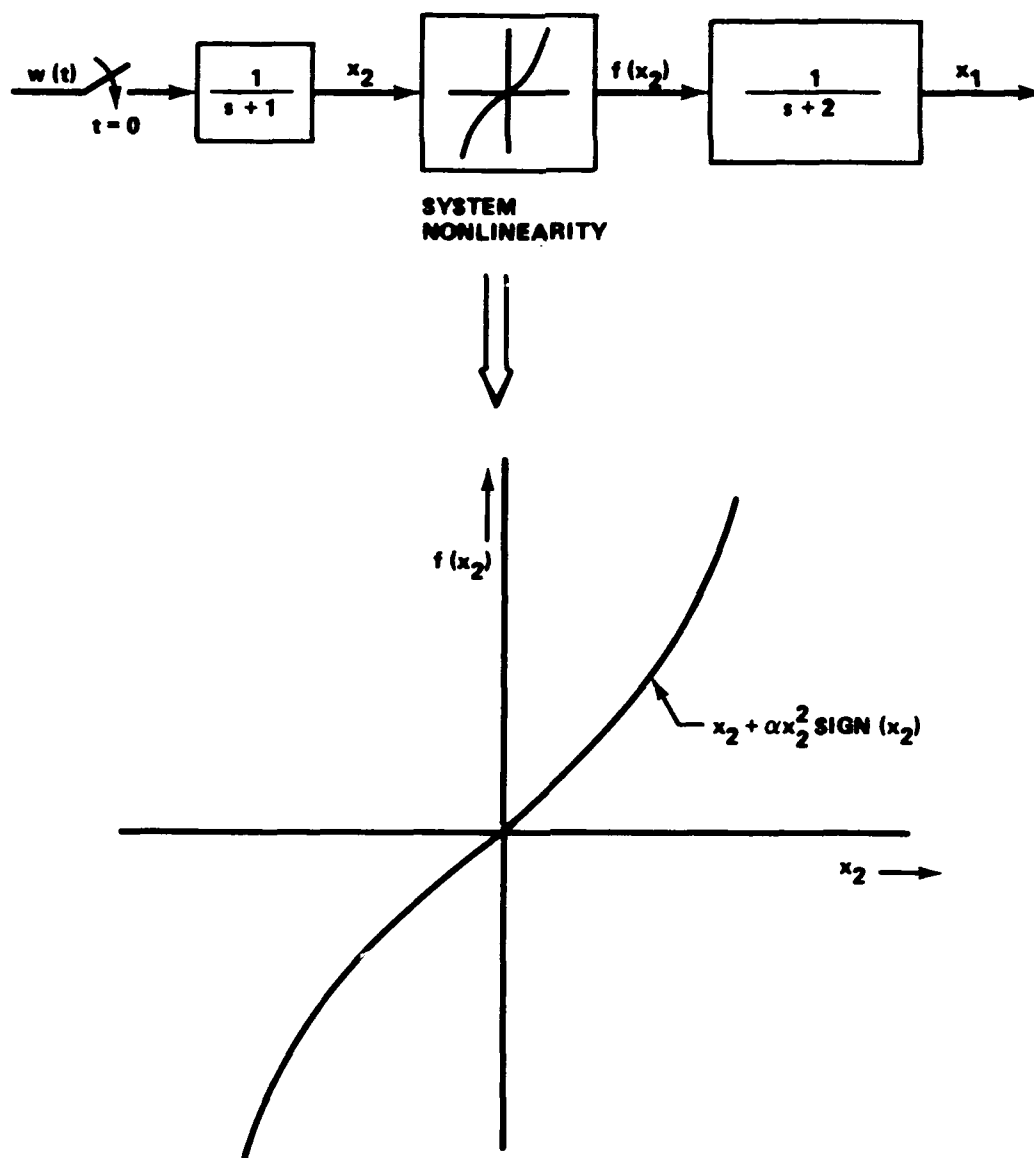


Figure 10. The Open-Loop System and System Nonlinearity
Described by Equation (24)

An interesting problem was encountered when varying $x_2(0)$ for this example. It was found that, for $\alpha = 0.05$ and $Q_w = 0.1$, the percent difference between the statistical covariance result and 1000 Monte Carlo runs increased rapidly as $x_2(0)$ was increased. However, for reasons explained later, it was suspected that the 1000 Monte Carlo simulation runs for large $x_2(0)$ were not giving correct results. To prove this

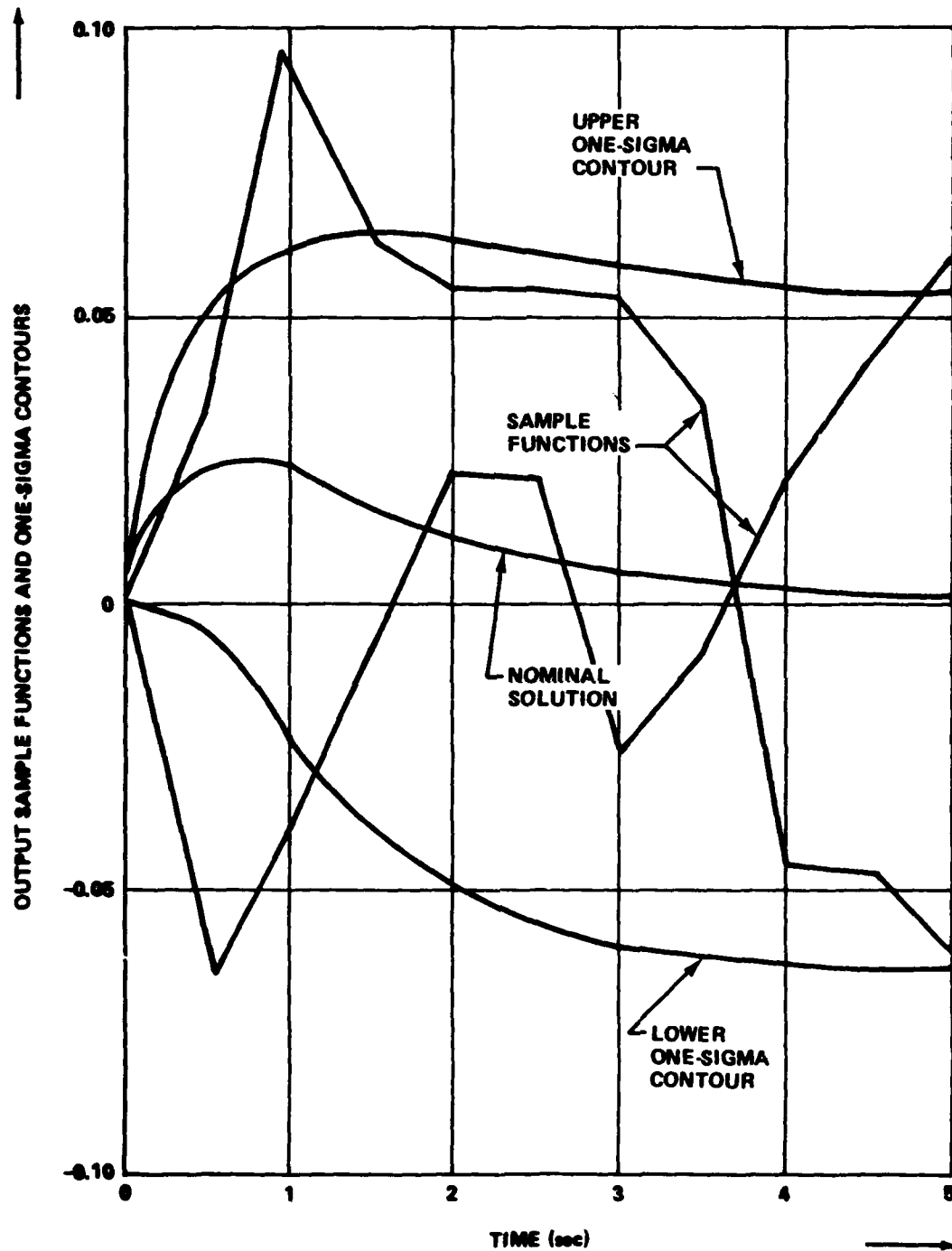


Figure 11. Monte Carlo Sample Functions and Statistical Covariance Results

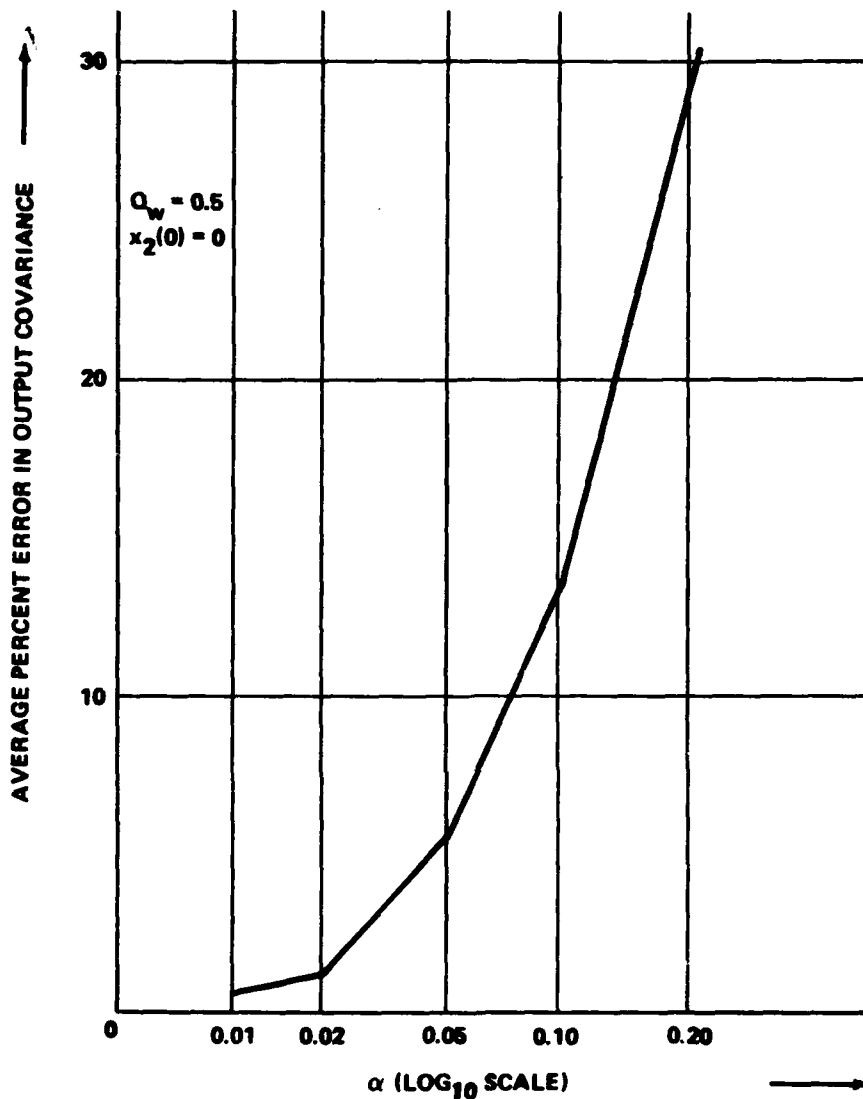


Figure 12. Variations in Statistical Covariance Results as a Function of the Amount of System Nonlinearity

conjecture, the linear system ($\alpha = 0$) was investigated by varying $x_2(0)$ as shown in Figure 16. After rechecking the computations to verify that step size and integration accuracy were acceptable, it was concluded that the inaccuracy of the digital random number generator must be the cause of the errors. In particular, the non-zero means and nonunity variance

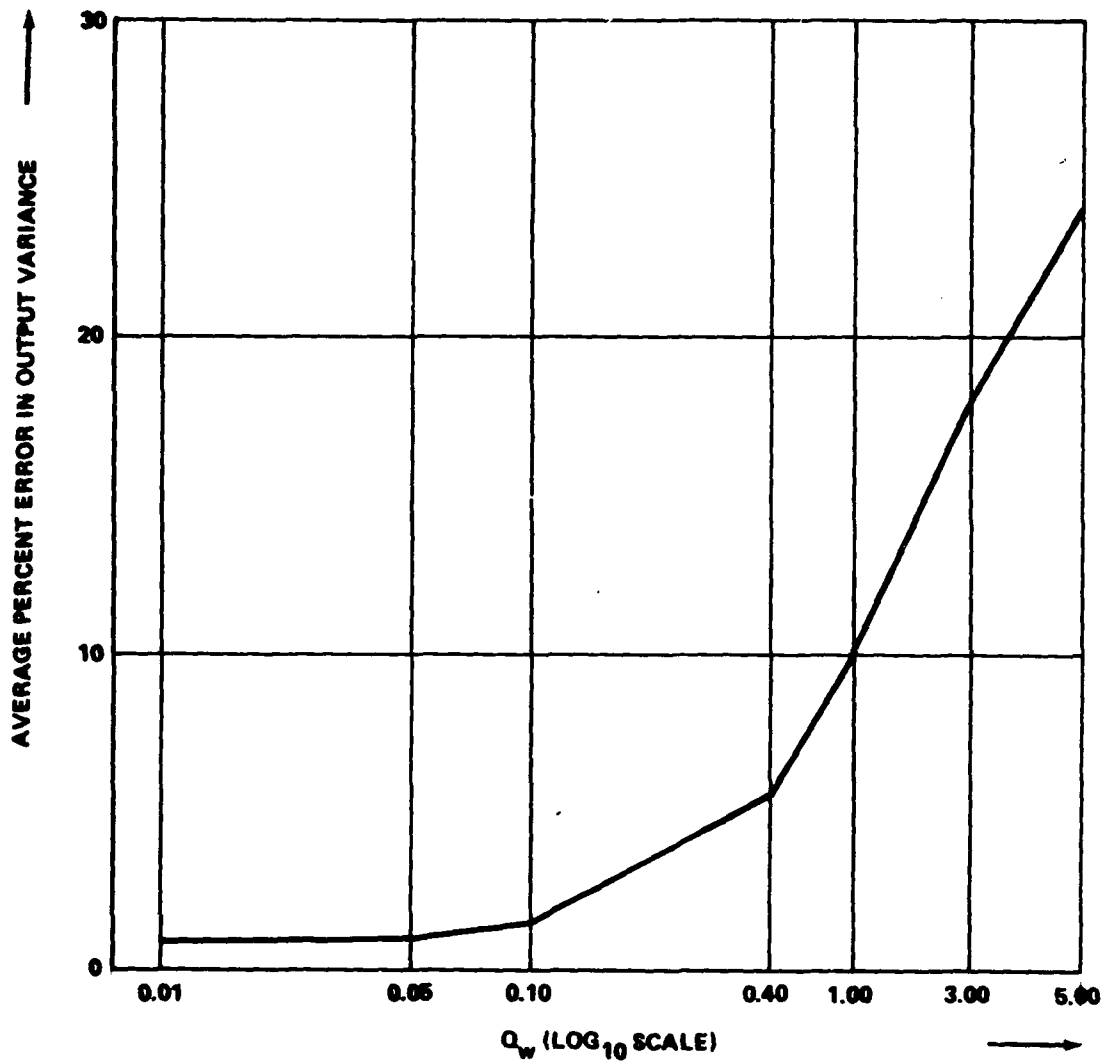


Figure 13. Variations in Statistical Covariance Results as a Function of the Input Noise

characteristics shown in Figures 2 and 3 of Paragraph 2 resulted in the unusual performance of Figure 16. The obvious conclusion is that a better random number sequence is needed for the realistic situation where the nominal trajectory is frequently far from zero.

(b) Example 2 — The statistical covariance technique is also valid for nonlinear feedback systems. Consider the system given in Figure 17, which is described by

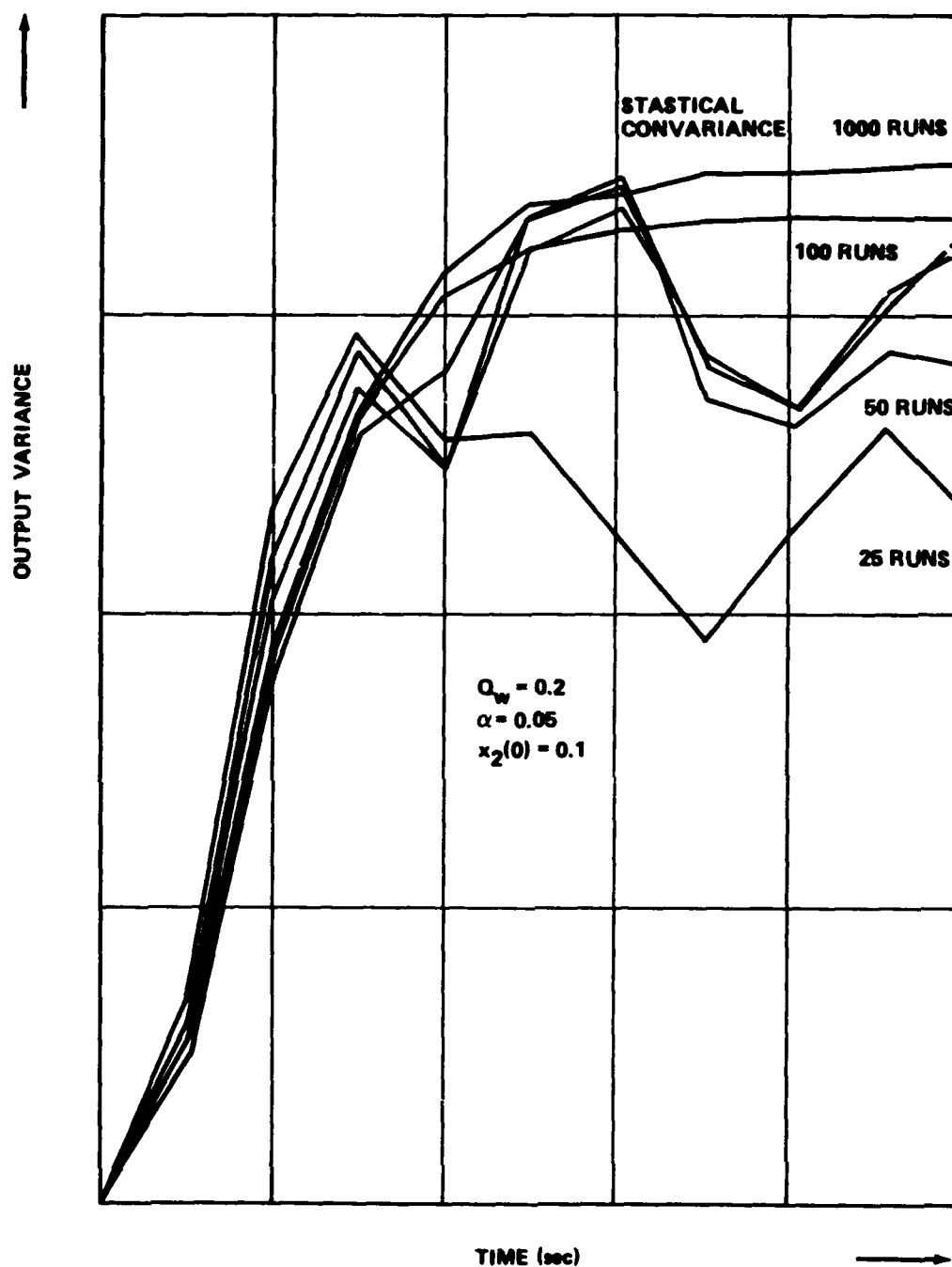


Figure 14. Monte Carlo and Statistical Covariance Results Versus Time

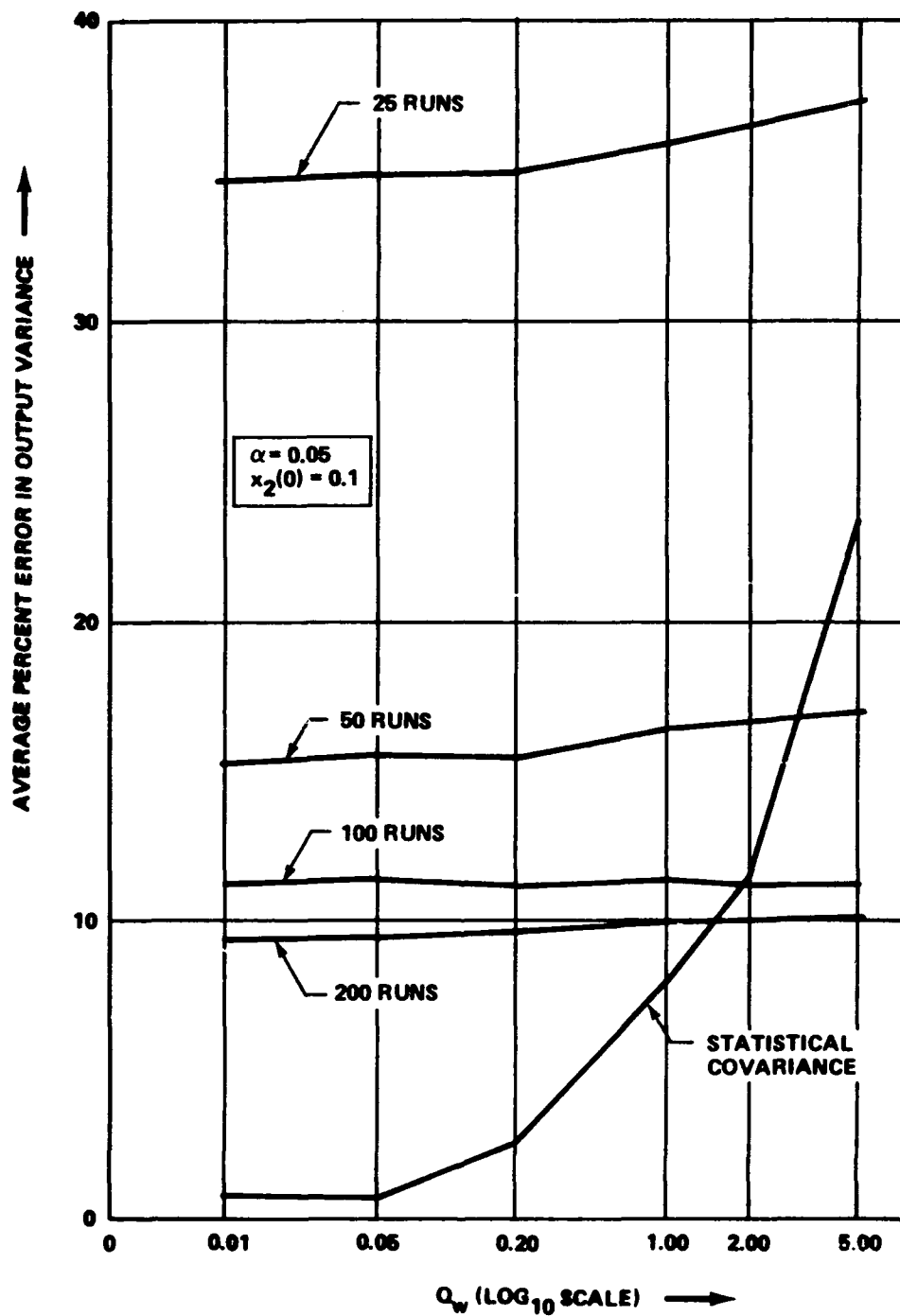


Figure 15. Variations in Average Percent Error in the Output Variance Versus Q_w

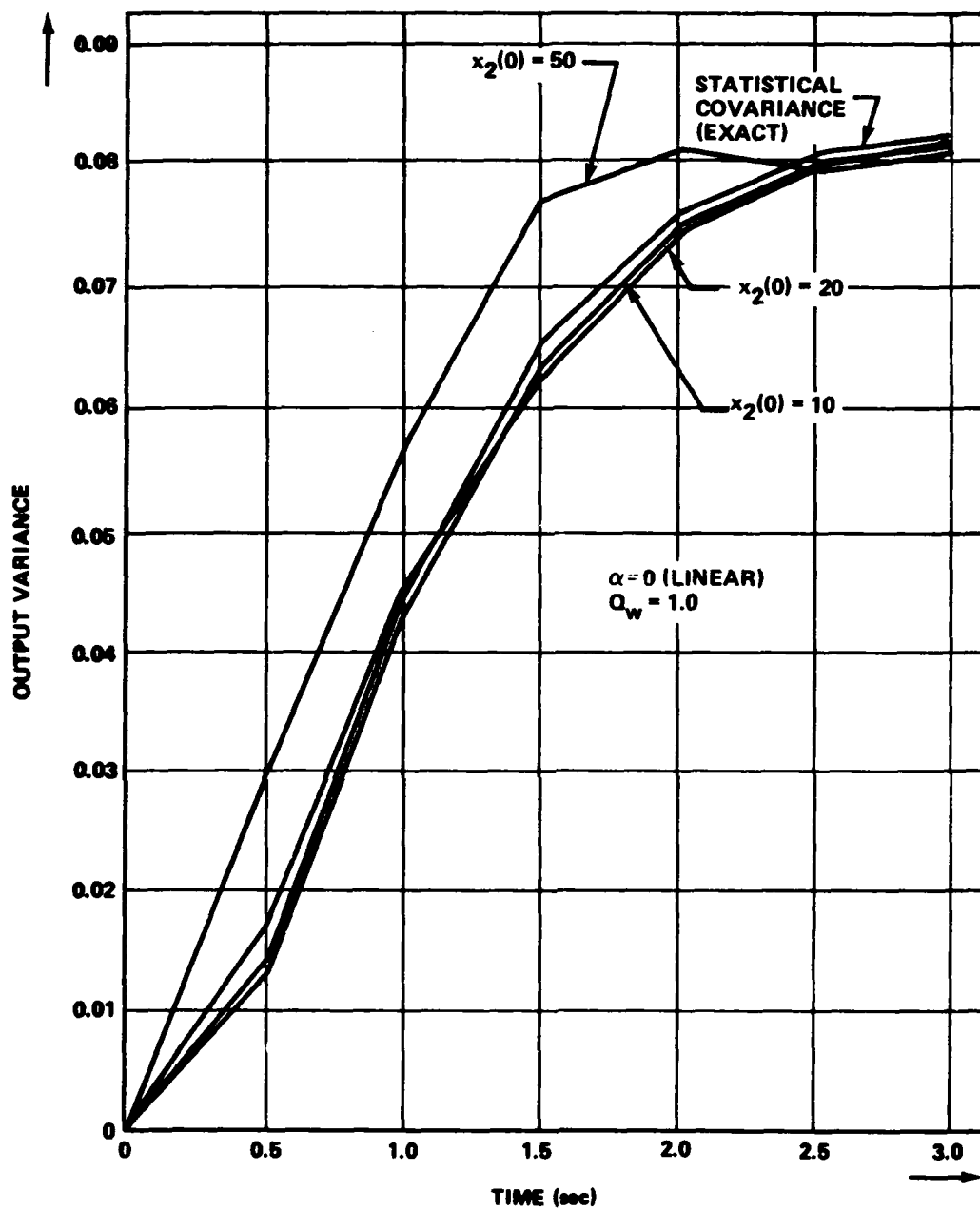


Figure 16. Time Plots of the Output Variance for Various $x_2(0)$

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + \alpha x_2^2 \operatorname{sign}(x_2) \\ \dot{x}_2 &= -x_1 - x_2 + w(t)\end{aligned}\quad (25)$$

Note that Equation (25) is the system Equation (24) of Figure 10 with the addition of a negative unity feedback loop. Results similar to those in Figure 15 for the open-loop case were obtained for Equation (25). These curves are shown in Figure 18.

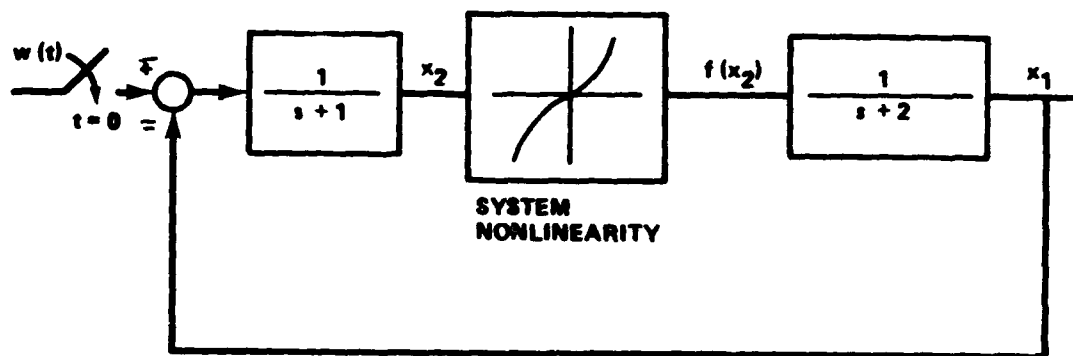


Figure 17. Closed-Loop System Description for Equation (25)

(2) Predicting Statistical Covariance Accuracy. It would be desirable to be able to predict in advance the accuracy of the statistical covariance technique for nonlinear systems. An exact prediction of the expected accuracy is not possible because no exact analytical solution can be found, in general, for the output variance of nonlinear systems. However, the result from a large number of Monte Carlo runs may be regarded as "exact" for the purpose of accuracy prediction, but even then (as shown in Paragraph 2) some inaccuracy is present. The reason for using the statistical covariance technique is to avoid the time-consuming Monte Carlo approach.

Suppose the Monte Carlo runs had been made for one particular design condition (parameter setting) of a given missile system. Using this information, the following procedure could be used to estimate the accuracy of the statistical covariance technique for sufficiently small changes in the parameter settings. As a particular example to illustrate the procedure, consider the exact incremental equation associated with Equation (24), i.e.,

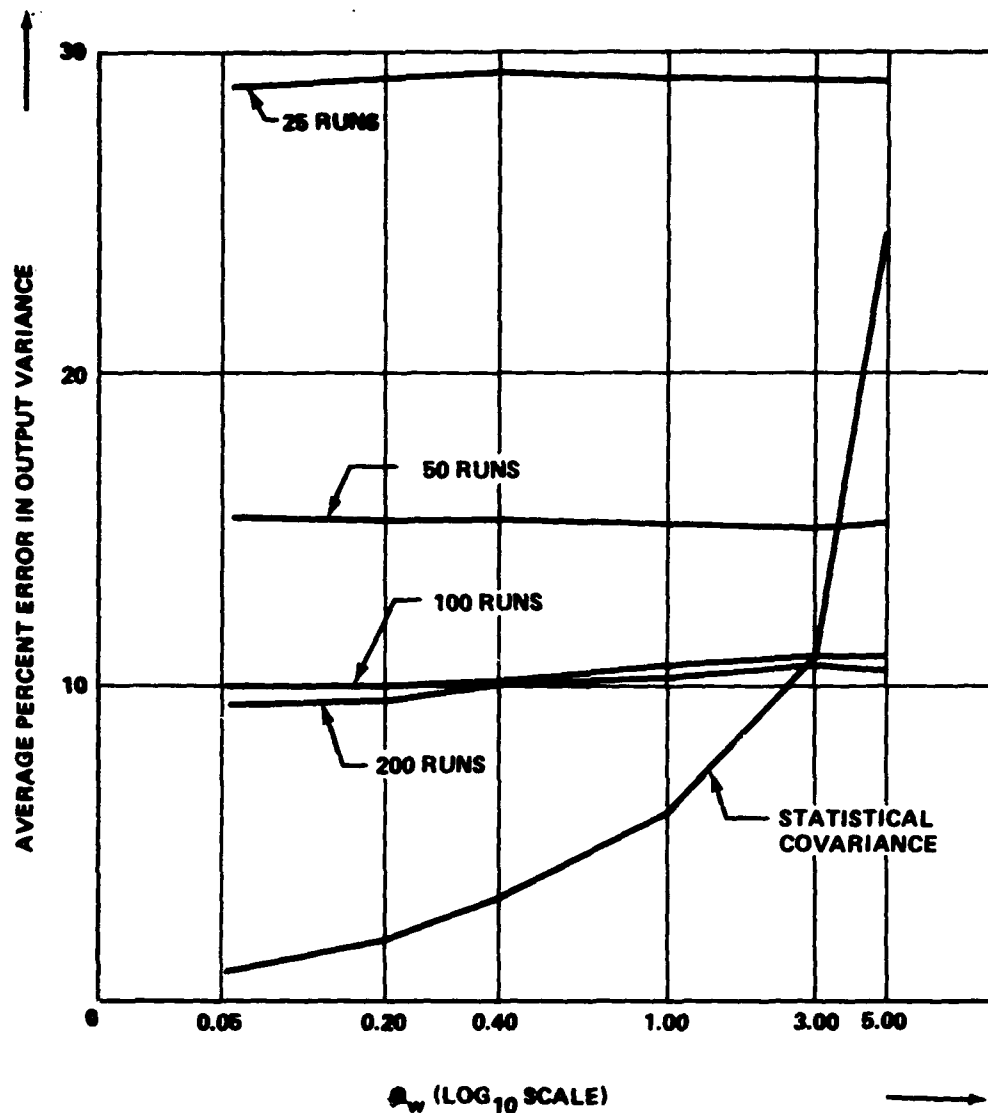


Figure 18. Monte Carlo and Statistical Covariance Results Versus Q_w for the Closed-Loop System for Equation (25)

$$\delta \dot{x}_1 = -2\delta x_1 + \left[1 + 2\alpha |x_2| \right]_N \delta x_2 + 2\alpha \delta x_2^2$$

$$\delta \dot{x}_2 = -\delta x_2 + w(t) \quad (26)$$

Suppose that the nonlinear term in Equation (26) is required to be not greater than k percent of the corresponding linear terms, i.e.,

$$\left| 2\alpha \delta x_2^2 \right| \leq \frac{k}{100} \left| -2\delta x_1 + \left[1 + 2\alpha \left| x_2 \right| \right]_N \delta x_2 \right| \quad (27)$$

Squaring Equation (27) and taking expected values yields

$$\begin{aligned} 4\alpha^2 \left(3\sigma_{\delta x_2}^4 \right) \leq \left(\frac{k}{100} \right)^2 & \left[4\sigma_{\delta x_1}^2 + \left[1 + 2\alpha \left| x_2 \right| \right]_N^2 \sigma_{\delta x_2}^2 \right. \\ & \left. + 4 \left[1 + 2\alpha \left| x_2 \right| \right]_N \left| E \left\{ \delta x_1 \delta x_2 \right\} \right| \right] \quad (28) \end{aligned}$$

Note that $E \left\{ \delta x_2^4 \right\}$ has been approximated by $3 \sigma_{\delta x_2}^4$, which is exact in this case because δx_2 is Gaussian. Using the steady-state values of the variance terms obtained from the linear case ($\alpha = 0$) yields

$$\begin{aligned} \sigma_{\delta x_1}^2 &= \frac{Q_w}{12} \\ \sigma_{\delta x_2}^2 &= \frac{Q_w}{2} \\ E \left\{ \delta x_1 \delta x_2 \right\} &= \frac{Q_w}{6} \quad (29) \end{aligned}$$

Substituting Equation (29) into Equation (28) gives, after simplifications,

$$3\alpha^2 Q_w \leq \left(\frac{k}{100} \right)^2 \left[\frac{1}{3} + \frac{2}{3} \left| 1 + 2\alpha x_2(t) \right|_N + \frac{1}{2} \left| 1 + 2\alpha x_2(t) \right|_N^2 \right] \quad (30)$$

The equality in Equation (30) is plotted in Figure 19, which shows that as either α or Q_w increases, the percent k of the first incremental equation in Equation (26) caused by the nonlinear term increases rapidly. Using the information in Figure 19 together with Figures 12 and 13, the percent error in the output variance as a function of the parameter k may be plotted. The sketch for varying α and Q_w is shown in Figure 20.

If k is less than 5 percent, then the error in the output variance is also less than 5 percent. However, for $k = 20$ percent, the error in the output variance is approximately 30 percent. If α and Q_w are such that k is

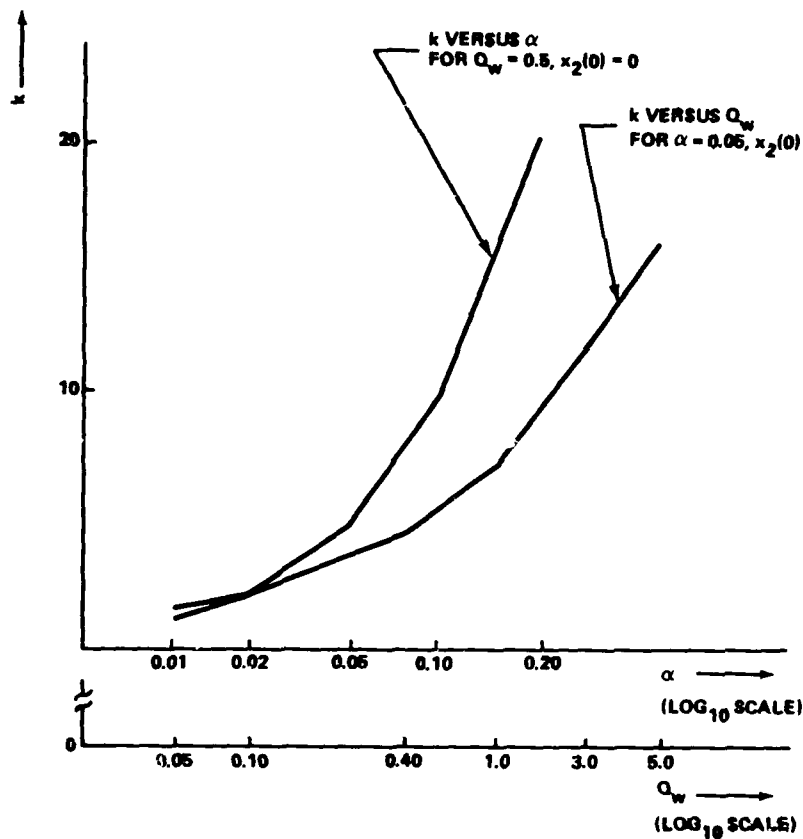


Figure 19. Plots of k Versus α and Q_w

approximately 20 percent, then the statistical covariance technique compares in accuracy to approximately 25 Monte Carlo runs (30 percent error). However, if $k = 5$ percent, then the accuracy of the statistical covariance technique is better than 200 Monte Carlo runs. Therefore, k may be computed in advance from the incremental equations to determine the expected accuracy and the number of Monte Carlo runs which would yield approximately the same accuracy as the statistical covariance technique for the given nonlinear system.

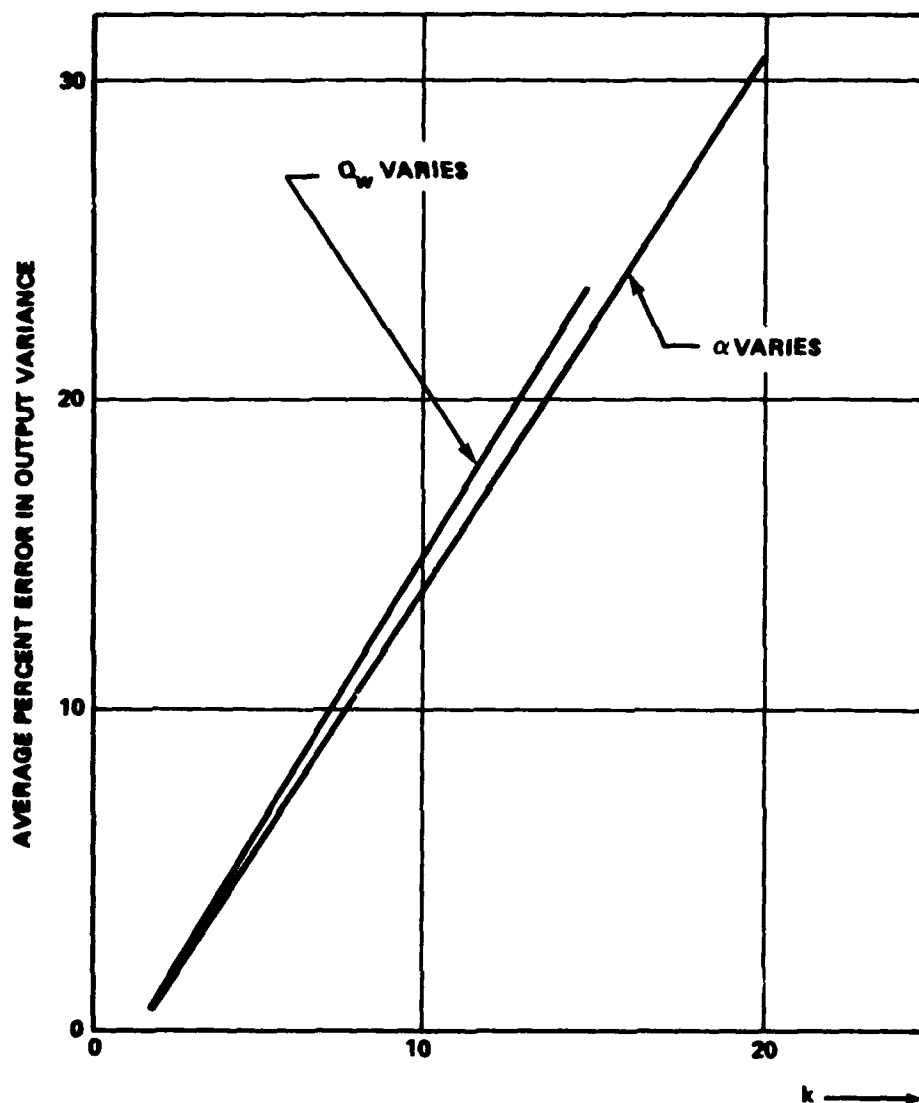


Figure 20. Plots of Percent Error for Statistical Covariance Versus k

These observations on the accuracy of the statistical covariance technique as a function of the quantity k are correct only for the single example previously considered. However, it can be expected that other second-order systems with parameters sufficiently near those of the previous example would yield results with corresponding accuracy. In particular, it should be expected, as shown in Figure 20, the average percent error in output variance would be on the order of 1.5 times the value of k . Moreover, some useful information would be obtained even if this error varied by as much as 1 to 2 times k . However, variations of from 50 to 100 times k would be unexpected.

Monte Carlo simulation experience is usually available on those missile systems where noise disturbances have been a problem. Curves similar to those in Figure 20 can be plotted for the particular missile system being considered. As stated previously, these curves can be used to yield approximate estimates of the accuracy of the statistical covariance technique in given situations.

c. Miss Distance Studies

It is important to be able to apply the results of the statistical covariance technique to miss distance studies in missile systems analysis. The only approximation made in extending the basic covariance method to nonlinear systems is that the incremental variations about the nominal flight conditions obey a linearized differential equation. As previously stated, this approximation is valid for sufficiently small noise disturbances or small nonlinear variations. However, to apply the statistical covariance technique to the missile miss distance problem, one further assumption is needed. This assumption is that the incremental system variables δx are Gaussianly distributed with a mean of zero and a covariance given by $P(t)$ from the statistical covariance solution.

There are several reasons to believe that the Gaussian assumption is a good one. First, recall that many of the noise input disturbances encountered in missile systems are approximately Gaussian. While Gaussian inputs to nonlinear systems do not result precisely in Gaussianly distributed system states, the deviation from the Gaussian condition is caused only by the inexactness of the approximation in linearizing the perturbation equations in δx . If these linearized equations were exact, then δx would be Gaussian for Gaussian inputs. A second reason for giving credence to the Gaussian assumption is the Central-Limit Theorem effect, which states that as the number of independent inputs becomes larger, the total effect on the system tends toward a Gaussian condition. Finally, experience has shown that the convolution property of linear subsystem equations inevitably makes the subsystem outputs much closer to being Gaussianly distributed than whatever input distribution was applied.

Applying the Gaussian assumption, the joint probability density function for the incremental state variation δx may be written as

$$p(\delta x) = \frac{1}{(2\pi)^n \sqrt{\det P(t)}} \exp \left[-\frac{1}{2} \delta x^T P^{-1} \delta x \right]. \quad (31)$$

If the statistical covariance result $P(t)$ is used only to compute the miss distance m_d , not all components of $P(t)$ are needed. In that case, a submatrix is formed by selecting from $P(t)$ those components included in

$$P_1(t) = \begin{pmatrix} \sigma_{\delta y}^2 & r\sigma_{\delta y}\sigma_{\delta z} \\ r\sigma_{\delta y}\sigma_{\delta z} & \sigma_{\delta z}^2 \end{pmatrix}, \quad (32)$$

where

$$\begin{aligned} \sigma_{\delta y}^2 &\triangleq E\{\delta y^2\} \\ \sigma_{\delta z}^2 &\triangleq E\{\delta z^2\} \\ r\sigma_{\delta y}\sigma_{\delta z} &\triangleq E\{\delta y\delta z\}. \end{aligned}$$

In Equation (32), δy and δz represent the incremental variations in orthogonal components of the miss distance (Figure 21). Therefore, if δy and δz are assumed to be jointly normal (Gaussian), the resulting joint density function is

$$p(\delta y, \delta z) = \frac{1}{2\pi \sqrt{\det P_1(t)}} \exp \left\{ -\frac{1}{2} (\delta y \ \delta z) P_1^{-1}(t) \begin{pmatrix} \delta y \\ \delta z \end{pmatrix} \right\}. \quad (33)$$

If Equation (31) holds, then so must Equation (33). However, Equation (33) may be true without Equation (31) being true, i.e., in miss distance studies using the statistical covariance method, it is necessary to assume only that δy and δz are jointly normal; the remaining system states may not be Gaussianly distributed.

In calculating the circular error probable (CEP) in miss distance studies, the probability density function of the miss distance, m_d , must be determined. The random variable m_d may be expressed either as

$$m_d = \sqrt{(\delta y)^2 + (\delta z)^2}, \quad (34)$$

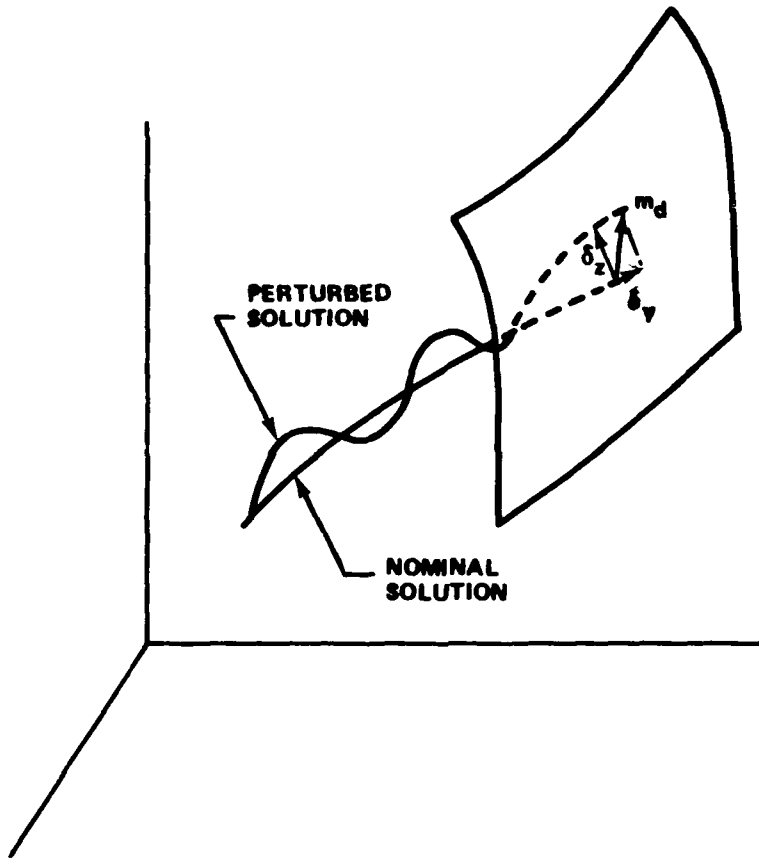


Figure 21. The Incremental Variation in the Miss Distance Resolved Into Components

which is the direct form used in Monte Carlo simulations or as

$$m_d = \sqrt{(\delta y^1)^2 + (\delta z^1)^2} \quad , \quad (35)$$

which is preferred for use with the statistical covariance technique.

In Equation (35), the uncorrelated variables δy^1 and δz^1 may be obtained by a linear transformation on δy and δz . As shown in detail in Appendix D, the resulting expression for the probability density function of m_d is

$$p(m_d) = \int_0^{2\pi} \frac{m_d}{2\pi \sigma_{\delta y^1} \sigma_{\delta z^1}} \exp \left[\frac{-m_d^2}{2} \left(\frac{\sin^2 \theta}{\sigma_{\delta y^1}^2} + \frac{\cos^2 \theta}{\sigma_{\delta z^1}^2} \right) \right] d\theta \quad , \quad (36)$$

where $\sigma_{\delta y}^2$ and $\sigma_{\delta z}^2$ are the variances of the transformed variables δy^1 and δz^1 , respectively. Moreover, the probability distribution function $P(m_d)$ is given by

$$P(m_d) = \int_0^{m_d} p(m_d) dm_d, \quad (37)$$

and the CEP is defined as that value of m_d for which

$$P(m_d) \Big|_{m_d} = \frac{1}{2}.$$

It should be pointed out that the definite integral in Equation (37) can be solved analytically only for the case in which

$$\sigma_{\delta y}^2 = \sigma_{\delta z}^2.$$

Otherwise, the integration may be performed numerically on the digital computer. If

$$\sigma_{\delta y}^2 = \sigma_{\delta z}^2,$$

then $p(m_d)$ in Equation (36) is the Rayleigh probability density function given by

$$p(m_d) = \begin{cases} \frac{m_d}{\sigma_{\delta y}^2} \exp \left[-\frac{m_d^2}{2\sigma_{\delta y}^2} \right] & \text{for } m_d \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (38)$$

Moreover, Equation (38) yields

$$P(m_d) = 1 - \exp \left[-\frac{m_d^2}{2\sigma_{\delta y}^2} \right]$$

from which

$$P(m_d) = \frac{1}{2}$$

when

$$m_d = \sigma_{\delta y1} \sqrt{2 \ln 2} = 1.17 \sigma_{\delta y1} .$$

Therefore, the CEP has been obtained analytically for this special case as $1.17 \sigma_{\delta y1}$. The probability density function $p(m_d)$ is shown in Figure 22. Whenever $\sigma_{\delta y1}^2 \neq \sigma_{\delta z1}^2$, the resulting probability density curve obtained via numerical integration has a similar shape, though not precisely a Rayleigh curve.

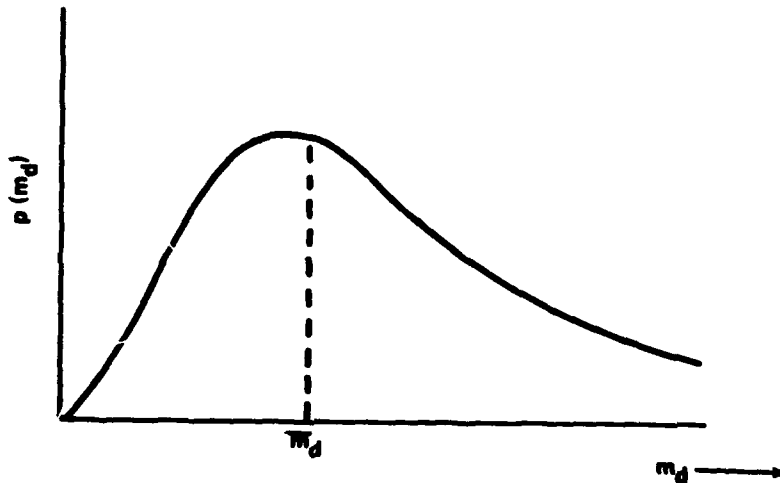


Figure 22. The Rayleigh Probability Density Function

In summary, results from the statistical covariance technique can be used to determine statistical information about missile systems miss distance. The procedure is to transform the correlated orthogonal components of the miss distance into an uncorrelated set of orthogonal

components δy^1 and δz^1 . The density and distribution functions of m_d are found from Equations (36) and (37) in terms of $\sigma_{\delta y^1}$ and $\sigma_{\delta z^1}$. Subsequently, the CEP can be determined by using the distribution function.

d. Error Budgeting

The statistical covariance technique can be applied to the problem of error budgeting for determining what part of the output variance is caused by any specific noise source. For example, if the noise source is the result of errors in a certain measuring device, such as a particular sensor, then it would be advantageous to know its effect on the missile miss distance. The term "error budgeting" refers to the synthesis problem; the allowable miss distance or "error" is given and the objective is to budget this allowable error among several contributing sources.

The statistical covariance technique uses linearized variational (or incremental) equations about a nominal flight path. As a result, the superposition principle can be applied using each independent input noise source individually with all others set to zero. Appropriate allocations can be made on this basis by adjusting weighting coefficients on the input noise sources until an acceptable combination is determined.

e. Summary

The statistical covariance technique has been developed in the paragraph for linear, time-varying systems. The practical use of the technique in an approximate analysis of nonlinear systems has been demonstrated on a second-order example. From the exact nonlinear incremental equations, the accuracy expected to be obtained by using the statistical covariance technique may be predicted for similar systems. Furthermore, in such cases, an estimate of the number of Monte Carlo runs needed to achieve this same accuracy may be determined.

In summary, the most dramatic result of Paragraph 3 is that not only does the statistical covariance technique yield useful results for nonlinear systems but an estimate of its accuracy and the number of Monte Carlo runs needed for comparable accuracy. In Paragraph 4, program refinements, such as an automatic program for calculating incremental equations, are described in detail. Subsequently, in Paragraph 5, a sequential algorithm is developed to apply statistical covariance more efficiently to large-scale missile systems.

4. Program Refinements for the Statistical Covariance Technique

a. Introduction

The use of the basic statistical covariance technique on simple, low-order, linear systems is straightforward. The method discussed in Paragraph 3 is much more satisfactory than the alternate approach (Monte Carlo) of ensemble-averaging a large number of simulation runs. Moreover, statistical covariance applies to a wide range of nonlinear problems, as described in Paragraph 3. If the combination of system nonlinearities and noise levels remain within acceptable limits, then the results of the statistical covariance technique are important in missile system analysis and design studies.

This paragraph examines the overall usefulness of the statistical covariance technique as a computer analysis tool for large-scale missile problems. Initially, the basic objectives are considered with special emphasis on their interdependence. Program refinements are suggested whenever possible to aid in realizing these overall objectives. Finally, an automatic interrogation "wrap-around" program is described for utilizing existing missile simulations in determining the required incremental equations for statistical covariance.

b. Computer Analysis Objectives

Basic objectives of computer analysis tools for large-scale systems may be conveniently grouped under the interrelated headings of accuracy, computational speed, computing equipment requirements, and ease of implementation. In this paragraph the statistical covariance technique will be examined from the viewpoint of how well it meets these analysis objectives.

(1) Accuracy. The primary objective of any computer analysis technique is to obtain sufficient accuracy for the results to be trusted. Without some degree of confidence in the accuracy of a given method, one cannot proceed with assurance to the more complicated design problem. In the case of the statistical covariance technique, the exact solution is obtained for linear systems. Furthermore, a very accurate, though not exact, solution is realized for mildly nonlinear systems with low noise levels. Only for highly nonlinear systems and/or high noise levels does the accuracy become unacceptable. Even in those cases, the statistical covariance results often compare favorably with the results of the usual 25 to 50 Monte Carlo runs. Moreover, using the approach developed in Paragraph 3, one can predict to some extent what accuracy may be expected by statistical covariance on a given problem. Because the statistical covariance technique may far exceed the acceptable level of accuracy, it is often possible to operate at a reduced accuracy to aid

in achieving one or more of the other, more elusive, interrelated goals, such as computational speed or core requirements of the digital computer. For example, if an error of only 1 percent is being achieved by statistical covariance, it might be advantageous to use either a larger step size or a faster, but less accurate, integration method to speed up the calculation while giving an error of 5 percent.

(2) Computational Speed. The second objective of a computer analysis software package is to achieve results at a satisfactory computational speed. As mentioned previously, speed is closely related to accuracy and may be improved if less accuracy is permitted. In addition to increasing speed by using a larger step size or a faster integration method, e.g., Euler's method, there are certain simplifying approximations which tend to speed up the basic statistical covariance algorithm. One of these is the use of constant coefficients in place of slowly time-varying coefficients in the incremental equations. These coefficients vary slowly if the nominal (noise-free) trajectory is relatively constant over a time span. Moreover, when particular coefficients, even if variable, are near zero, neglecting them entirely can often increase computational speed significantly. This approximations will be utilized in the automatic sensitivity program.

(3) Computing Equipment Requirements. The third objective of computer-oriented tools is that they be amenable for operating on a small or medium-sized digital computer, i.e., the limited equipment aspect of all engineering problems must be considered in introducing a new computer analysis tool. One consideration is word-length of the computer. The statistical covariance technique can be easily handled on the 32-bit Sigma 5. Work-length problems would arise in this connection only if the technique were being used on much smaller bit machines, e.g., minicomputers. A second, much more important consideration is the amount of core required for processing data in the statistical covariance technique. A sequential algorithm is described in Paragraph 5 to aid in reducing excessive core requirements. A rapid access device (RAD) may be used as an extended core for overlaying problems. An RAD makes available 1.5 million words of core in addition to the 32 K words of the main memory. It should be noted that using the RAD decreases the computational speed of a given algorithm. In particular, there is a delay caused by an average latency time of 17 msec and a data transfer from the RAD to the main memory (50 msec required to transfer 1500 words). Therefore, recognizing that only limited equipment is available is important in the design of computer-oriented algorithms. The statistical covariance technique can be tailored for operating on limited computing equipment.

(4) Ease of Implementation. A fourth objective of computer analysis techniques is that the algorithms be easily applied from the user's viewpoint to a given problem. Special programming or the need for precalculated data detracts from any computer method. For the

statistical covariance technique, the structural form of the coefficients for the incremental (or sensitivity) equations may be determined in advance by taking partial derivatives as needed. An alternate approach based on automatic machine computation will follow.

c. An Automatic Sensitivity Computer Package

The three basic parts of the statistical covariance package are the determination of the incremental equation coefficients, the integration of the covariance equations, and the integration of the nominal system equations. The purpose of this section is to describe how the incremental equation coefficients may be automatically computed from an existing simulation program of the noise-free system.

Recall that for the nonlinear system given by Equation (2), i.e.,

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{w}, t) \quad ,$$

the incremental equation coefficients are given by Equation (23), i.e.,

$$A(t) = \frac{\partial \underline{f}}{\partial \underline{x}} \left| \begin{array}{l} \underline{x} = \underline{x}_N(t) \\ \underline{w} = \underline{w} \end{array} \right.$$

$$B(t) = \frac{\partial \underline{f}}{\partial \underline{w}} \left| \begin{array}{l} \underline{x} = \underline{x}_N(t) \\ \underline{w} = \underline{w} \end{array} \right.$$

The partial derivatives in Equation (23) may be computed numerically as

$$A(t) = \frac{\underline{f}(\underline{x}_N + \Delta \underline{x}, \underline{w}, t) - \underline{f}(\underline{x}_N, \underline{w}, t)}{\Delta \underline{x}}$$

$$B(t) = \frac{\underline{f}(\underline{x}_N, \underline{w} + \Delta \underline{w}, t) - \underline{f}(\underline{x}_N, \underline{w}, t)}{\Delta \underline{w}} \quad , \quad (39)$$

where $\eta_{\underline{w}}$ represents the mean value of the input disturbance vector \underline{w} , which previously has been assumed to be zero. The notations $\Delta \underline{x}$ and $\Delta \underline{w}$ in Equation (39) represent small perturbations about the nominal trajectory $\underline{x}_N(t)$. Because $P(t)$ is being computed within the same time advance loop, $\Delta \underline{x}$ and $\Delta \underline{w}$ may be selected as one-tenth of the standard deviation of $\underline{x}(t)$ and $\underline{w}(t)$, respectively. Whenever any main diagonal element of $P(t)$ is near zero, a lower limit on $\Delta \underline{x}$, such as 10^{-4} , should be utilized.

This numerical evaluation of partial derivatives may also be applied for estimating the accuracy expected in the analysis of nonlinear systems. Equations may be formed similar to Equation (39) by using $\Delta \underline{x}$ and $\Delta \underline{w}$ as one-sigma values, rather than one-tenth of a sigma in each case, i.e.,

$$\begin{aligned} \underline{f}(\underline{x}_N + \sigma_{\underline{x}}, \eta_{\underline{w}}, t) - \underline{f}(\underline{x}_N, \eta_{\underline{w}}, t) &= A(t) \sigma_{\underline{x}} + \underline{g}_1(\underline{x}_N, \eta_{\underline{w}}, t) \\ \underline{f}(\underline{x}_N, \eta_{\underline{w}} + \sigma_{\underline{w}}, t) - \underline{f}(\underline{x}_N, \eta_{\underline{w}}, t) &= B(t) \sigma_{\underline{w}} + \underline{g}_2(\underline{x}_N, \eta_{\underline{w}}, t) \end{aligned} \quad (40)$$

The vectors \underline{g}_1 and \underline{g}_2 in Equation (40) represent the higher order terms in the Taylor series expansion of \underline{f} about $(\underline{x}_N, \eta_{\underline{w}})$, which in this case has only been approximated numerically. A vector \underline{k} may be computed whose components k_i represent the percent of the incremental equations caused by the higher order terms, where

$$k_i = \frac{\underline{g}_{1i}(\underline{x}_N, \eta_{\underline{w}}, t) + \underline{g}_{2i}(\underline{x}_N, \eta_{\underline{w}}, t)}{\left| A(t) \sigma_{\underline{x}} + B(t) \sigma_{\underline{w}} \right|_i} \cdot 100 \text{ percent}$$

These components of \underline{k} enable the results of Paragraph 3 to be used to obtain an estimate of the accuracy to be expected from the statistical covariance technique.

Although these numerical procedures for determining $A(t)$ and $B(t)$ are yet to be verified in actual nonlinear system simulations, they represent a realistic approach toward achieving a self-contained computer software package. The nominal system equations of existing simulations can be placed inside the complete package which will interrogate the system equations as required with small perturbations to form $A(t)$ and $B(t)$. It will not even be necessary to know what the system equations

are; only their number and the number and location of noise inputs are required. Such a comprehensive computer software package for statistical covariance would be extremely useful for a wide variety of missile systems analysis problems.

d. Summary

Program refinements for statistical covariance have been described from the viewpoint of their impact on basic computer analysis objectives: accuracy, computational speed, equipment required, and ease of implementation. Numerical procedures have been developed for determining the coefficients of the incremental equations. Furthermore, an estimate of the accuracy of the statistical covariance technique may be obtained. Finally, the philosophy of a self-contained computer software package has been advanced to virtually eliminate the majority of implementation problems.

5. Sequential Calculations of the Statistical Covariance Equations

a. Introduction

In large-scale air defense missile systems it is convenient to be able to perform computations sequentially to avoid excessive digital computer storage requirements. Mesarovic [35, 36] proposed the concept of multilevel systems, whereby complex systems are partitioned into simpler subsystems for analysis and design purposes. Mesarovic used a hierarchy of system models to form a stratified description of complex systems. Lefkowitz [37] described how the multilevel hierarchy approach has been used to solve particular industrial problems. Noton [38] has applied multilevel systems theory to derive a coordination algorithm for a number of subsystem Kalman estimators.

A sequential algorithm, related to the multilevel system concept, is developed in this paragraph for applying the statistical covariance technique to large-scale missile systems. As shown in Figure 23, the overall system is segmented into several subsystems interconnected by feed-forward and feedback paths. It is believed that large-scale systems can be handled more efficiently by the sequential algorithm than by the basic covariance program.

b. The Sequential Approach

Consider again the basic statistical covariance equation developed in Equation (17) of Paragraph 3 and repeated here for convenience.

$$\dot{P} = AP + P^T A^T + BQB^T$$

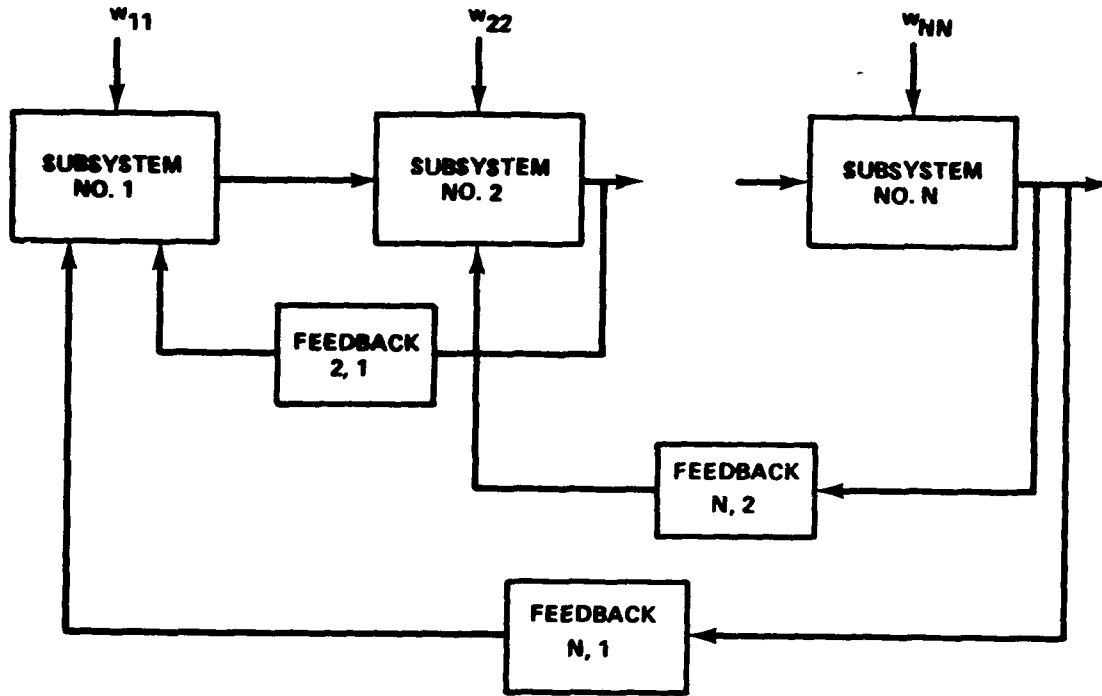


Figure 23. The Basic Large-Scale System Showing Individual Subsystem Connections

Equation (17) describes the evolution of the state covariance matrix $P(t)$ as time increases from t_0 . Let the large-scale system be segmented into several subsystems as shown in Figure 23. In the subsystem context, the matrices A , B , P , and Q may be partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & & & & A_{2N} \\ \vdots & & & & & \vdots \\ A_{N1} & A_{N2} & \cdot & \cdot & \cdot & A_{NN} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & \cdot & \cdot & \cdot & \cdot & B_{1N} \\ B_{21} & & & & & \vdots \\ \vdots & & & & & \vdots \\ B_{N1} & \cdot & \cdot & \cdot & \cdot & B_{NN} \end{pmatrix}$$

$$P = \begin{pmatrix} P_{11} & P_{12} & . & . & . & P_{1N} \\ P_{21} & P_{22} & . & . & . & P_{2N} \\ \vdots & & & & & \vdots \\ P_{N1} & P_{N2} & . & . & . & P_{NN} \end{pmatrix} \quad Q = \begin{pmatrix} Q_{11} & . & . & . & Q_{1N} \\ \vdots & & & & \vdots \\ Q_{N1} & . & . & . & Q_{NN} \end{pmatrix}$$

The following two assumptions are made at this point:

- 1) The matrix P is symmetric, i.e., $P = P^T$ and in particular,

$$P_{ij} = P_{ji}^T .$$

- 2) Disturbances are uncorrelated with each other and each enters only the single designated subsystems as shown in Figure 23. This means that B and Q are diagonal.

Therefore, Equation (17) may be expressed as

$$\begin{aligned} \dot{P}_{ij} = & A_{i1}P_{1j} + A_{i2}P_{2j} + A_{i3}P_{3j} + \dots + A_{iN}P_{Nj} \\ & + P_{1i}^T A_{j1}^T + P_{2i}^T A_{j2}^T + P_{3i}^T A_{j3}^T + \dots + P_{Ni}^T A_{jN}^T \\ & + B_{ii} Q_{ii} B_{ii}^T \delta_{ij} , \end{aligned} \quad (41)$$

where δ_{ij} is zero if $i \neq j$ and unity if $i = j$. Equation (41) may be conveniently grouped as

$$\begin{aligned} \dot{P}_{ij} = & \sum_{k=1}^{i-1} A_{ik} P_{kj} + A_{ii} P_{ij} + \sum_{k=i+1}^N A_{ik} P_{kj} \\ & + \sum_{k=1}^{j-1} P_{ki}^T A_{jk}^T + P_{ji}^T A_{jj}^T + \sum_{k=j+1}^N P_{ki}^T A_{jk}^T + B_{ii} Q_{ii} B_{ii}^T \delta_{ij} \end{aligned}$$

for $i = 1 \dots, N$ and $j = 1 \dots, N$. (42)

The matrices P_{ki}^T and P_{ji}^T in the second line of Equation (42) may be replaced by P_{ik} and P_{ij} , respectively, because P is symmetric. The second and fifth terms in Equation (42) are the only ones involving P_{ij} . Moreover, the first and fourth terms have as coefficient matrices entries from the lower left of the main diagonal of the system matrix A . These terms represent feed-forward paths. Elements from the upper right of the main diagonal of A appear in the third and sixth terms in Equation (42) and represent feedback paths. The seventh, or last, term represents input noise data. Rewriting Equation (42) and summarizing these observations, yields

$$\begin{aligned} \dot{P}_{ij} = & \underbrace{A_{ii} P_{ij} + P_{ij} A_{jj}^T}_{\text{Self-Interaction}} + \underbrace{\left[\sum_{k=1}^{i-1} A_{ik} P_{kj} + \sum_{k=1}^{j-1} P_{ik} A_{jk}^T \right]}_{\text{Feed-Forward Paths}} \\ & + \underbrace{\left[\sum_{k=i+1}^N A_{ik} P_{kj} + \sum_{k=j+1}^N P_{ik} A_{jk}^T \right]}_{\text{Feedback Paths}} + \underbrace{B_{ii} Q_{ii} B_{ii}^T \delta_{ij}}_{\text{Noisy Inputs}} . \end{aligned} \quad (43)$$

What is desired is to apply Equation (43) sequentially to determine P_{ij} for all i and all j . It is particularly important only to know P_{ii} , but it may be shown that calculation for all i and j is necessary to completely determine P_{ii} .

For example, consider the problem of determining P_{44} for the case where no feedback terms are present. Using Equation (43) yields

$$\begin{aligned}\dot{P}_{44} &= A_{44} P_{44} + P_{44} A_{44}^T + A_{43} P_{34} + P_{34} A_{43}^T + B_{44} Q_{44} B_{44}^T \\ \dot{P}_{34} &= A_{33} P_{34} + P_{34} A_{44}^T + A_{32} P_{24} + P_{33} A_{43}^T \\ \dot{P}_{24} &= A_{22} P_{24} + P_{24} A_{44}^T + A_{21} P_{14} + P_{23} A_{43}^T \\ \dot{P}_{14} &= A_{11} P_{14} + P_{14} A_{44}^T + P_{13} A_{43}^T\end{aligned}\quad (44)$$

In interpreting Equation (44), note that the four matrix equations must be applied sequentially in reverse order, beginning with the last. The matrices P_{13} , P_{23} , and P_{33} are known from calculations for the previous subsystem, i.e., No. 3. It is assumed that subsystem No. 4 has inputs only from subsystem No. 3 and from external sources.

A flow chart describing the sequential method for statistical covariance calculations is provided in Figure 24. Additional investigation is required to develop a workable computer software package for use in large-scale missile system analysis.

c. Summary

This paragraph has described a sequential algorithm as a means of handling large-scale analysis problems more efficiently than the basic statistical covariance program. The new algorithm categorizes terms on a subsystem basis as self-interacting, feed-forward paths, feedback paths, and noisy inputs. Further work is needed to develop the sequential algorithm into a useable computer software package.

6. Conclusions and Recommendations

a. Conclusions

The primary conclusion of this report is that the statistical covariance technique yields significantly faster, more accurate, results than the traditional Monte Carlo approach for a wide range of missile systems analysis error propagation problems. The statistical covariance algorithm, which gives exact results for linear, time-varying

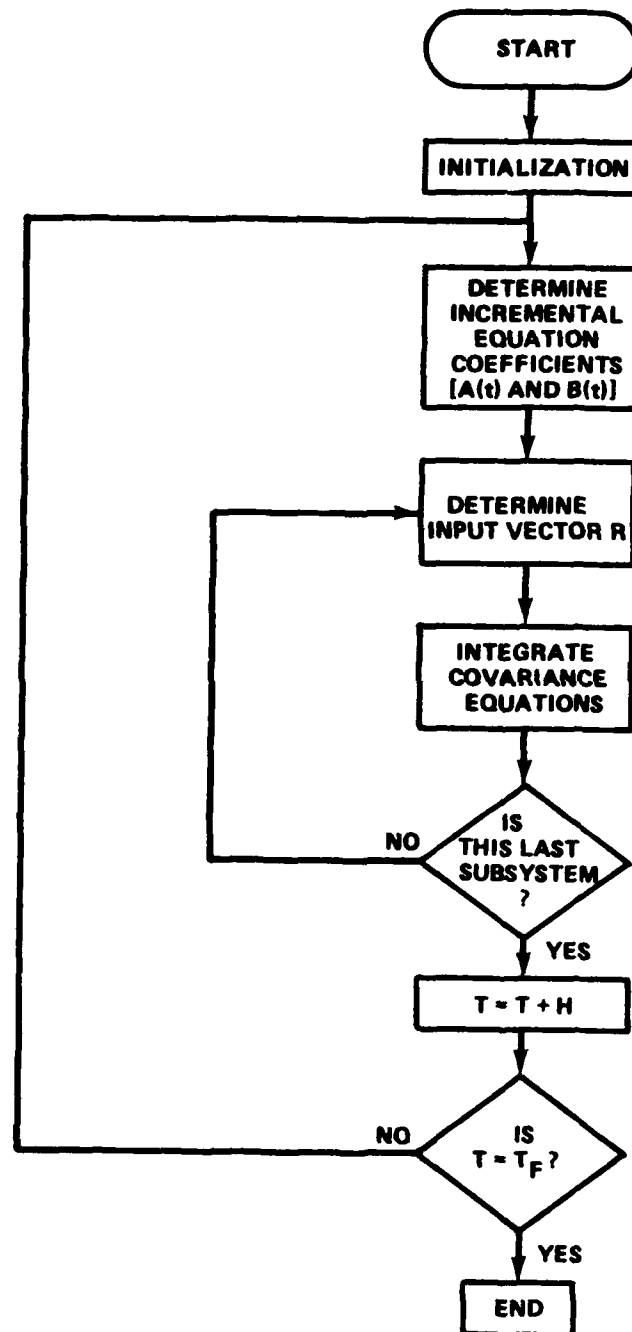


Figure 24. A Flow Chart of the Sequential Algorithm

systems, may be applied to nonlinear systems by using linearized incremental equations about the noise-free solution. It was stated in Paragraph 3 that the accuracy of the statistical covariance technique for nonlinear systems depends entirely upon the relative accuracy of this linearizing approximation. A parameter (α) of a second-order nonlinear example was varied to demonstrate that, as a system becomes highly nonlinear, a larger error is obtained. Moreover, as the input noise level (Q_w) was increased to very high values, the error in the statistical covariance results was increased. It may be concluded that, while there admittedly are extreme conditions for which the statistical covariance algorithm yields unacceptable errors, the technique works satisfactorily for the conditions occurring in most missile system analysis problems.

A special interrogation program has been suggested for automatically computing the incremental equation coefficients by using existing missile system simulations. This "wrap-around" program interrogates an existing simulation program to determine locally linear sensitivity coefficients. Therefore, it is unnecessary to reprogram a complex missile system simulation to apply the statistical covariance technique. Furthermore, the automatic sensitivity program is capable of computing information required for estimating the accuracy of the statistical covariance results and the number of Monte Carlo runs needed for comparable accuracy.

Procedures for transforming the statistical covariance results to determine miss distances were also described in Paragraph 3. A short integration program is required for handling the results of the transformation for a complete missile systems analysis package.

A sequential algorithm was developed in Paragraph 3 to aid in reducing the amount of required computer storage. The new algorithm treats the overall problem on a subsystem basis and performs statistical covariance calculations serially from subsystem to subsystem. An additional approximation on sampling in feedback loops must be made to permit the sequential ordering of these calculations.

It has been demonstrated that the Monte Carlo approach often yields poor results in error propagation studies. In Paragraph 2 effort was devoted to obtain a correct implementation of the Monte Carlo method. In particular, several pseudo-random number generators were tested on a second-order linear system. It was noted that, even for the best of these generators, 1000 Monte Carlo runs only yielded an average error of 1.5 percent for the output variance. Furthermore, for the same linear system errors of 30, 15, and 10 percent were obtained for 25, 50, and 100 runs, respectively. Similar results were obtained for a second-order nonlinear system. The conclusion is that a larger number of Monte Carlo runs than usually performed are required for acceptable accuracy. Rather than expending excessive computer time for large numbers of Monte Carlo runs, a more satisfactory approach is to utilize the faster statistical covariance technique in a totally preprogrammed computer software package.

It has been shown that the statistical covariance technique is applicable to a wide range of missile system analysis problems. In Paragraph 4, the technique was evaluated according to four general objectives of all computer analysis tools: accuracy, computational speed, equipment requirements, and ease of implementation. Extreme accuracy was demonstrated in Paragraphs 2 and 3 on linear and nonlinear systems. The accuracy prediction scheme may become important for highly nonlinear systems with high noise levels. The computational speed was compared favorably with Monte Carlo in Paragraph 4. The sequential algorithm of Paragraph 5 was developed to help in solving the computer storage problem, and the use of RAD further aids in making equipment requirements less critical. Finally, the implementation problem is greatly reduced by using the automatic sensitivity program for coefficient calculations as an integral part of the computer software package.

b. Recommendations for Further Work

The first recommendation is that a total computer software package for statistical covariance be developed especially tailored for missile systems analysis problems. In addition to the basic algorithm, this package should include the automatic sensitivity program for calculating incremental equation coefficients, an accuracy prediction program associated with the automatic sensitivity program, and a miss distance transformation program. A comprehensive self-contained package containing these programs would require only minimal information from the user while providing the flexibility of performing difficult missile system analysis production runs.

The second recommendation is that the sequential version of the statistical covariance technique be programmed and tested as a means of reducing computer storage requirements. Only when it has been shown that a significant savings in storage is realized by the sequential algorithm should it be incorporated into the total computer software package.

Finally, the third recommendation is that improvements be made in Monte Carlo simulations. The major deficiency at present is the need for a better random number sequence to serve as system inputs. In addition, more work on the discretization problem of continuous random signals could be useful.

The development of these improvements in statistical techniques would greatly enhance missile system analysis capabilities. Better analysis techniques, as those described herein, must be further developed and tested in practical systems operations.

Appendix A. COMPUTER PROGRAMS FOR PSEUDO-RANDOM NUMBER GENERATION

This appendix provides additional computer programs and output data on the generation and testing of pseudo-random numbers from multiplicative formulas. Figure A-1 gives the output of the computer program of the Brown, ($M = 2^{20}$) generator shown in Figure 1. Corresponding results for the Brown ($M = 2^{31}$) generator are included in Figures A-2 and A-3. Figure A-4 shows the computer program used to obtain information for the $M = 2^{20}$ curves in Figures 2 and 3. The program in Figure A-5 was used for the Monte Carlo results plotted in Figures 5 and 6. These detailed computer programs have been included in this Appendix for use in the exact reproduction of the two Brown multiplicative generators described in Paragraph 2 and their use in Monte Carlo simulations.

	Unnormalized Integers IX	Uniformly Distributed Numbers U	Normally Distributed Numbers XNORM
1	630504441	.295892	-.610237
2	-1039920661	.745625	-.042897
3	-2038644671	.886779	.580360
4	399654211	.139956	.312584
5	1445011913	.070748	1.790409
6	447652699	.914882	1.980184
7	-2049858543	.897445	.337223
8	1903303219	.131396	.315461
9	408017049	.115380	1.508073
10	955525067	.259715	-.126775
11	251416929	.769868	.204443
12	230720035	.031772	.708874
13	-790089623	.488181	-.2619232
14	829984571	.534968	-1.168766
15	1343072177	.853440	.676851
16	423683347	.055926	.528589
17	224549817	.904595	1.982828
18	-1650800213	.325765	-.205215
19	37910273	.154054	.849193
20	1101817987	.606204	-1.519265
21	-911736055	.499259	-1.200526
22	-1914386661	.701390	-.354427
23	1582862161	.534989	-.021977
24	380018771	.272449	-.157236
25	941694425	.049787	1.460078
26	-1082427509	.283315	-.479490
27	-599381471	.077971	1.401332
28	1968000395	.832004	1.113074
29	-207854551	.056235	.569028
30	76466930	.924556	2.134694

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Figure A-1. The First 30 Numbers from the Brown ($M = 2^{20}$) Generator in Figure 1

```

1      C      OPERATION OF A MULTIPLICATIVE DIGITAL GENERATOR
2      C
3      C
4      XMEAN=0.
5      SIG=1.
6      IX=31571
7      DUM=0.1
8      DO 99 K=1,30
9      C
10     C      MULTIPLICATIVE GENERATOR
11     C      WITH M=2 TO THE 31ST POWER
12     C
13     IY=19971*IX
14     IX=IY
15     IF (IX) 5,5,6
16     5 IY=IY+2147483647+1
17     6 U=IY*0.4656612873E-9
18     C
19     C      TRANSFORMATION TO NORMAL
20     C
21     Z=SQRT (-2.0*ALOG(DUM))*SIG
22     XNORM=Z*COS(6.28318*U)+XMEAN
23     C
24     DUM=U
25     PRINT 77,K,IX,U,XNORM
26     77 FORMAT(6X,I4,8X,I12,2F15.6)
27     99 CONTINUE
28     STOP
29     END

```

Figure A-2. The Brown Multiplicative Pseudo-Random Number Generator
with $M = 2^{31}$

	Unnormalized Integers IX	Uniformly Distributed Numbers U	Normally Distributed Numbers XNORM
1	630504441	.293601	-.580569
2	-1039920661	.515749	-1.557927
3	-2028644671	.027399	1.133758
4	399654211	.186103	1.048162
5	1445011913	.672886	-.854173
6	447652699	.208454	.229733
7	-2049858543	.045460	1.699142
8	1903303219	.886295	1.978327
9	408017049	.189998	.180880
10	955525067	.444951	-1.714557
11	251414929	.117075	.943558
12	230720035	.107437	1.616939
13	-790089623	.632086	-1.425650
14	829984571	.386492	-.724391
15	1343072177	.625417	-.972457
16	423683347	.197293	.315021
17	294549817	.137160	1.173032
18	-1650800213	.231286	.233840
19	37910273	.017653	1.700681
20	1191817987	.554983	-2.673539
21	-911736055	.575440	-.965546
22	-1914386661	.108544	.916140
23	1522862161	.737078	-.170929
24	340918771	.177379	.744172
25	941694425	.438511	-1.722732
26	-1022427509	.495955	-1.283618
27	-589381471	.725548	-.181241
28	1948000995	.916422	.693097
29	-297854551	.861301	.268936
30	76466939	.035608	.532845

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Figure A-3. The First 30 Number from the Brown ($M = 2^{31}$) Generator in Figure A-2 of Appendix A


```

1      C      RANDOM NUMBER GENERATOR TESTING
2      C
3      C
4      XMEAN=0.
5      SIG=1.
6      IX=31571
7      DUM=0.1
8      SMEAN=0.
9      SVAR=0.
10     DR 88 L=1,20
11     XL=L
12     DR 99 K=1,500
13
14     C      MULTIPLICATIVE GENERATOR
15     C      WITH M=2 TO THE 20TH POWER
16     C
17     IV=19971*IX
18     IYP=IV/1048576
19     IX=IV-IYP*1048576
20     AX=IX
21     U=AX/1048576.
22     IF(U)5,5,6
23     5 U=-U
24     6 CONTINUE
25     IX=IV
26
27     C      TRANSFORMATION TO NORMAL
28     C
29     Z=SQRT(-2.0*ALOG(DUM))*SIG
30     XNORM=Z*COS(6.28318*U)+XMEAN
31     C
32     DUM=U
33     SMEAN=SMEAN+XNORM
34     SVAR=SVAR+XNORM*XNORM
35     99 CONTINUE
36     ZL=XL*500.
37     ZMEAN=SMEAN/ZL
38     ZVAR=SVAR/ZL
39     PRINT 8,ZL,ZMEAN,ZVAR
40     8 FORMAT(15X,F7.1,10X,F6.4,10X,F6.4)
41     88 CONTINUE
42     STOP
43     END

```



Figure A-4. A Digital Computer Program for Testing the Normally Distributed Brown ($M = 2^{20}$) Generator

```

1      C      MONTE CARLO RUN FOR A SECOND-ORDER
2      C      LINEAR SYSTEM IN PHASE VARIABLE FORM
3      C
4      C
5      DIMENSION XE(2),XS(2),XMO(2),XM1(2),S(10,2),
6      1 SOL(10,2),DIF(10,2)
7      T=0.
8      H=.05
9      MS=2
10     NUM=25
11     XNUM=NUM
12     NTOTAL=100
13     MTHT=NTOTAL/10
14     DO 31 K=1,MS
15     DO 31 N=1,MTHT
16     31 S(N,K)=0.
17     XMEAN=0.
18     SIG=1.0/SQRT(H)
19     IX=31571
20     DUM=0.1
21     DO 32 M=1,NUM
22     XE(1)=0.
23     XE(2)=0.
24     DO 42 N=1,MTHT
25     DO 52 L=1,10
26     IY=19971+IX
27     IYP=IY/1048576
28     IX=IY-IYP*1048576
29     AX=IX
30     U=AX/1048576.
31     IF(U)5,5,6
32     5 U=-U
33     6 CONTINUE
34     IX=IY
35     Z=SQRT(-2.0*ALOG(DUM))*SIG
36     XNORM=Z*COS(6.28318*U)+XMEAN
37     DUM=U
38     CALL XEQN(XE,XMO,XNORM)
39     DO 23 K=1,MS
40     23 XS(K)=XE(K)+H*XMO(K)
41     CALL XEQN(XS,XM1,XNORM)
42     DO 24 K=1,MS
43     24 XE(K)=XE(K)+0.5*H*(XMO(K)+XM1(K))
44     52 CONTINUE
45     DO 53 K=1,MS
46     53 S(N,K)=S(N,K)+XE(K)*XF(K)
47     42 CONTINUE
48     32 CONTINUE
49     KA=1
50     DO 62 NA=1,MTHT
51     XNA=NA

```

Figure A-5. A Monte Carlo Simulation Program for the System
in Equation (7) Using the Brown ($M = 2^{20}$) Generator

Appendix B. THE EXACT CONVOLUTION SOLUTION FOR THE OUTPUT VARIANCE IN LINEAR SYSTEMS

Consider a linear system with zero input for $t < 0$ and with a stationary random process $w(t)$ as input for $t \geq 0$. Let $w(t)$ be a zero-mean white noise process with variance Q_w . Because the input statistics are different for the two ranges of t , the effective input must be considered to be nonstationary. Papoulis [39] has shown that the expression for the output autocorrelation is given by

$$R_{xx}(t_1, t_2) = R_{ww}(t_1, t_2) * h(t_1) * h(t_2) \quad , \quad (B-1)$$

where $*$ represents the convolution operation and $h(t)$ is the impulse response of the linear system, i.e., the inverse Laplace transform of the transfer function $H(s)$.

As an example, consider the second-order system given in Equation (7) of Paragraph 1. The impulse response $h(t)$ is obtained by using Equation (8), i.e.,

$$\begin{aligned} h(t) &= L^{-1} H(s) = L^{-1} \left\{ \frac{1}{(s+1)(s+2)} \right\} \\ &= e^{-t} - e^{-2t} \quad t \geq 0 \quad . \end{aligned} \quad (B-2)$$

Performing the first convolution in Equation (B-1) yields

$$\begin{aligned} R_{ww}(t_1, t_2) * h(t_1) &= Q_w \delta(t_1 - t_2) * (e^{-t_1} - e^{-2t_1}) \\ &\quad \text{for } t_1 \geq 0, \quad t_2 \geq 0 \\ &= \int_0^\infty Q_w \delta(t_1 - t_2 - \tau) [e^{-\tau} - e^{-2\tau}] d\tau \\ &= \begin{cases} Q_w [e^{-(t_1 - t_2)} - e^{-2(t_1 - t_2)}]; & t_1 \geq t_2 \geq 0 \\ 0 & \text{Otherwise} \end{cases} \end{aligned} \quad (B-3)$$

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The second convolution yields

$$\begin{aligned}
 R_{xx}(t_1, t_2) &= [R_{ww}(t_1, t_2) * h(t_1)] * h(t_2) \\
 &= \int_0^{t_2} Q_w \left[e^{-(t_1 - t_2 + \tau)} - e^{-2(t_1 - t_2 + \tau)} \right] \cdot \left[e^{-\tau} - e^{-2\tau} \right] d\tau \\
 R_{xx}(t_1, t_2) &= Q_w e^{-(t_1 - t_2)} \left[\frac{1}{2} (1 - e^{-2t_2}) - \frac{1}{3} (1 - e^{-3t_2}) \right] \\
 &\quad - Q_w e^{-2(t_1 - t_2)} \left[\frac{1}{3} (1 - e^{-3t_2}) - \frac{1}{4} (1 - e^{-4t_2}) \right] \\
 &\quad \text{for } t_1 \geq t_2 \geq 0 \quad . \quad (B-4)
 \end{aligned}$$

Therefore, the output variance $Q_x(t)$ is given by

$$\begin{aligned}
 Q_x(t) &= R_{xx}(t_1, t_2) \Big|_{t_1 = t_2 = t} \\
 Q_x(t) &= Q_w \left[\frac{1}{12} - \frac{1}{2} e^{-2t} + \frac{2}{3} e^{-3t} - \frac{1}{4} e^{-4t} \right] \\
 &\quad \text{for } t \geq 0 \quad . \quad (B-5)
 \end{aligned}$$

This exact solution is used for comparison with Monte Carlo simulations in Figure 5.

Appendix C. COMBINED STATISTICAL COVARIANCE AND MONTE CARLO COMPUTER PROGRAMS

The combined statistical covariance and Monte Carlo simulation program for a second-order nonlinear system (Figure C-1) is included in this appendix. This basic program was modified for different input data to produce the numerical results given in Figures 11 through 20.

```

1      C      A COMBINED PROGRAM FOR STATISTICAL COVARIANCE
2      C      AND MONTE CARLO FOR A NONLINEAR SYSTEM
3      C
4      C
5      DIMENSION P1(3),PS(3),PM0(3),PM1(3),Q(1,1),A(2,2),
6      1 XE(2),XMO(2),XM1(2),XS(2),B(2),R(3),S(10),
7      2 SOL(10),DIF(10),Z(10)
8      COMMON ALPHA
9      ALPHA=.05
10     Q(1,1)=.1
11     AXE2=.1
12     NUM=1000
13     XNUM=NUM
14     NTOTAL=100
15     H=.05
16     MS=2
17     NS=3
18     MTOT=10
19     LT=NTOTAL/MTOT
20     XLT=LT
21     P(1)=0.
22     P(2)=0.
23     R(3)=Q(1,1)
24     XE(1)=0.
25     XE(2)=AXE2
26     DO 2 K=1,NS
27     2 P1(K)=0.
28     A(1,1)=.2.
29     A(2,2)=.1.
30     A(2,1)=0.
31     DO 11 N=1,MTOT
32     DO 12 L=1,LT
33     A(1,2)=+(1+.2.*ALPHA*ABS(XE(2)))
34     C*****
35     C      INTEGRATION USING RK2
36     CALL DER1(P1,PM0,A,R)
37     DO 13 K=1,NS
38     13 PS(K)=P1(K)+ H * PM0(K)
39     CALL DER1(PS,PM1,A,R)
40     DO 14 K=1,NS
41     14 P1(K)=P1(K)+ 0.5*H*(PM0(K)+PM1(K))
42     C*****
43     C      INTEGRATION USING RK2
44     XNORM=0.
45     CALL XEQNT(XE,XMO,XNORM)
46     DO 23 K=1,MS
47     23 XS(K)=XE(K)+H*XMO(K)
48     CALL XEQN(XS,XM1,XNORM)
49     DO 24 K=1,MS
50     24 XE(K)=XE(K)+0.5*H*(XMO(K)+XM1(K))
51     12 CONTINUE

```

Figure C-1. A Combined Program for Statistical Covariance and Monte Carlo for a Nonlinear System

```

52      SOL(N)=PI(1)
53      11 CONTINUE
54      DO 31 N=1,MTOT
55      Z(N)=0.
56      31 S(N)=0.
57      XMEAN=0.
58      SIG=SQRT(Q(1,1)/H)
59      IX=31571
60      DUM=0.1
61      DO 32 M=1,NUM
62      XE(1)=0.
63      XE(2)=AXE2
64      DO 42 N=1,MTOT
65      DO 52 L=1,LT
66      IY=19971+IX
67      IYP=IY/1048576
68      IX=IY-IYP*1048576
69      AX=IX
70      U=AX/1048576.
71      IF(U)5,5,6
72      5 U=-U
73      6 CONTINUE
74      IX=IY
75      Y=SQRT(-2.0*ALOG(DUM))*SIG
76      XNORM=Y*COS(6.28318*U)+XMEAN
77      DUM=U
78      C*****
79      C      INTEGRATION USING RK2
80      CALL XEQN(XE,XM0,XNORM)
81      DO 73 K=1,MS
82      73 XS(K)=XE(K)+H*XM0(K)
83      CALL XEQN(XS,XM1,XNORM)
84      DO 74 K=1,MS
85      74 XE(K)=XE(K)+0.5*H*(XM0(K)+XM1(K))
86      52 CONTINUE
87      Z(N)=Z(N)+XE(1)
88      S(N)=S(N)+XE(1)*XE(1)
89      42 CONTINUE
90      32 CONTINUE
91      DO 62 NA=1,MTOT
92      XNA=NA
93      T=H*XNA*XLT
94      Z(NA)=Z(NA)/XNUM
95      S(NA)=S(NA)/XNUM
96      ST(NA)=S(NA)-Z(NA)*Z(NA)
97      DIF(NA)=100.*(S(NA)-SOL(NA))/S(NA)
98      PRINT 7,T,S(NA),SOL(NA),DIF(NA)
99      7 FORMAT(10X,F5.2,3F15.10)
100     62 CONTINUE
101     SS1=0.
102     DO 97 NA=1,MTOT
103     97 SS1=SS1+ABS(DIF(NA))
104     SI=SS1*0.1

```

Figure C-1. Continued


```

105      PRINT 94751
106      94 FORMAT(20X,F20.8)
107      PRINT 547,NUM,Q(1,1),ALPHA,AXE2
108      547 FORMAT(10X,I6,10X,3F8.4)
109      STOP
110      END

```

```

1      SUBROUTINE DER1(PD,PMD,A,R)
2      DIMENSION PD(3),PMD(3),A(2,2),R(3)
3      PMD(1)=2.*A(1,1)*PD(1)+2.*A(1,2)*PD(2)+R(1)
4      PMD(2)=A(2,1)*PD(1)+(A(1,1)+A(2,2))*PD(2)
5      1+A(1,2)*PD(3)+R(2)
6      PMD(3)=2.*A(2,1)*PD(2)+2.*A(2,2)*PD(3)+R(3)
7      RETURN
8      END

```

```

1      SUBROUTINE XEGN(XD,XMD,RT)
2      DIMENSION XD(2),XMD(2)
3      COMMON ALPHA
4      IF(XD(2))211,212,212
5      211 AXY=-1.
6      GOTO 213
7      212 AXY=+1.
8      213 CONTINUE
9      XMD(1)=2.*XD(1)+XD(2)+ALPHA*AXY*XD(2)*XD(2)
10     XMD(2)=XD(2)+RT
11     RETURN
12     END

```

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.50	.0014649311	.0014606304	.2935711145
1.00	.0044823773	.0044761673	.1385438442
1.50	.0065570101	.0065642931	.1110710502
2.00	.0077750459	.0076042712	2.1964464188
2.50	.0082906336	.0080520175	2.8781394958
3.00	.0083774291	.0082301907	1.7575597763
3.50	.0084621906	.0082975477	1.9456300735
4.00	.0084444605	.0083219223	1.4862840652
4.50	.0085169263	.0083303005	2.1912345886
5.00	.0084073506	.0083329417	.8850463033
		1.40835190	
1000	.1000	.0500	.1000

```

*STOP= 0
FIN
TOTAL JOB TIME=00:03:38
BEGIN IDLE

```

Figure C-1. Concluded

Appendix D. A TRANSFORMATION FOR NORMAL RANDOM VARIABLES

The objective of this appendix is to develop a linear transformation for transforming two correlated normal random variables, δy and δz , into two uncorrelated normal random variables, δy^1 and δz^1 . The results of this transformation are expressed in Equation (32) of Paragraph 3.

Assuming zero means for all variables, consider the linear transformation defined by

$$\begin{aligned}\delta y^1 &= \delta y \cos \alpha - \delta z \sin \alpha \\ \delta z^1 &= \delta y \sin \alpha + \delta z \cos \alpha\end{aligned}\quad (D-1)$$

The requirement for δy^1 and δz^1 to be uncorrelated is that $E\{\delta y^1 \delta z^1\} = 0$, i.e.,

$$\begin{aligned}E\{\delta y^1 \delta z^1\} &= E\{(\delta y)^2 \cos \alpha \sin \alpha - \delta y \delta z \sin^2 \alpha \\ &\quad + \delta y \delta z \cos^2 \alpha - (\delta z)^2 \cos^2 \alpha\}\end{aligned}\quad (D-2)$$

The value of α for which Equation (D-2) equals zero is given by

$$\alpha = \frac{1}{2} \tan^{-1} \frac{2E\{\delta y \delta z\}}{\sigma_{\delta z}^2 - \sigma_{\delta y}^2}\quad (D-3)$$

Moreover, the values of $\sigma_{\delta y^1}^2$ and $\sigma_{\delta z^1}^2$ are

$$\begin{aligned}\sigma_{\delta y^1}^2 &= \sigma_{\delta y}^2 \sin^2 \alpha + \sigma_{\delta z}^2 \cos^2 \alpha \\ &\quad + 2r_{\delta y \delta z} \sigma_{\delta y} \sigma_{\delta z} \cos \alpha \sin \alpha \\ \sigma_{\delta z^1}^2 &= \sigma_{\delta y}^2 \cos^2 \alpha + \sigma_{\delta z}^2 \sin^2 \alpha \\ &\quad - 2r_{\delta y \delta z} \cos \alpha \sin \alpha\end{aligned}\quad (D-4)$$

Therefore, Equation (D-3) completely specifies the required transformation, and Equation (D-4) may be used in Equation (32) of Paragraph 3 to yield the required density function.

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